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RECURSIVE REDUCED-ORDER OPEN-LOOP OPTIMAL CONTROL OF DISCRETE WEAKLY COUPLED LINEAR SYSTEMS

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SUMMARY

A recursive reduced-order algorithm with an arbitrary degree of accuracy is obtained for solving the open-loop optimal control problem of discrete weakly coupled linear systems. The original two-point boundary value problem is transformed into two reduced-order initial value problems via the use of a non-singular transformation. The solutions by the proposed algorithm converge to the optimal solutions of discrete time open-loop control with a rate of convergence of $O(\varepsilon^2)$, where ε is a small coupling parameter. The obtained results are in the spirit of the parallel reduced-order algorithms for small-parameter optimal control problems. The algorithm produces considerable savings in required computations.

KEY WORDS weak coupling; decomposition; optimal control; parallel algorithms

1. INTRODUCTION

The study of linear weakly coupled control systems was originated by Kokotovic *et al.*[1] The recursive approach to linear weakly coupled systems, based on fixed point iterations, has been developed recently.[2] It has been shown that the recursive method is particularly useful when the coupling parameter ε is not extremely small and/or when any desired order of accuracy is required, namely $O(\varepsilon^{2j})$, where $j = 1, 2, 3, \ldots$. The recursive method has been applied successfully to the reduced-order solution of closed- and open-loop control problems of weakly coupled continuous systems.[3] The optimal reduced-order solution of the weakly coupled algebraic discrete Riccati equation has been obtained.[4]

In this short communication we extend the results[2–4] to the open-loop optimal control problem of discrete weakly coupled systems. The solution of this problem will be obtained in terms of reduced-order difference equations.

A weakly coupled linear time-invariant discrete system is represented by[2]

$$\begin{aligned} \mathbf{x}_1(k+1) &= \mathbf{A}_1\mathbf{x}_1(k) + \varepsilon\mathbf{A}_2\mathbf{x}_2(k) + \mathbf{B}_1\mathbf{u}_1(k) + \varepsilon\mathbf{B}_2\mathbf{u}_2(k) \\ \mathbf{x}_2(k+1) &= \varepsilon\mathbf{A}_3\mathbf{x}_1(k) + \mathbf{A}_4\mathbf{x}_2(k) + \varepsilon\mathbf{B}_3\mathbf{u}_1(k) + \mathbf{B}_4\mathbf{u}_2(k) \\ \mathbf{x}_1(0) &= \mathbf{x}_{10}, \qquad \mathbf{x}_2(0) = \mathbf{x}_{20} \end{aligned} \tag{1}$$

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with substates $\mathbf{x}_i \in \mathbb{R}^{n_i}$ and subcontrol inputs $u_i \in \mathbb{R}^{m_i}$ respectively, $i = 1, 2$; ε is a small coupling parameter, i.e. we assume that subsystem matrices $\varepsilon\mathbf{A}_2$, $\varepsilon\mathbf{B}_2$, $\varepsilon\mathbf{A}_3$ and $\varepsilon\mathbf{B}_3$ are in some sense 'smaller' than subsystem matrices $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{A}_4$ and $\mathbf{B}_4$. The performance criterion of the corresponding linear–quadratic control problem is defined by

$$J = \frac{1}{2}\mathbf{x}^{\mathrm{T}}(n)\mathbf{F}\mathbf{x}(n) + \frac{1}{2}\sum_{k=0}^{n-1}[\mathbf{x}^{\mathrm{T}}(n)\mathbf{Q}\mathbf{x}(k) + \mathbf{u}^{\mathrm{T}}(k)\mathbf{R}\mathbf{u}(k)] \tag{2}$$

where

$$\mathbf{x}(k) = \begin{bmatrix} \mathbf{x}_1(k) \\ \mathbf{x}_2(k) \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \varepsilon\mathbf{Q}_2 \\ \varepsilon\mathbf{Q}_2^{\mathrm{T}} & \mathbf{Q}_3 \end{bmatrix} \geqslant \mathbf{0}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_1 & \varepsilon\mathbf{F}_2 \\ \varepsilon\mathbf{F}_2^{\mathrm{T}} & \mathbf{F}_3 \end{bmatrix} \geqslant \mathbf{0}, \quad \mathbf{R} = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix} > \mathbf{0} \tag{3}$$

The open-loop optimal control problem has the solution given by

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1) \tag{4}$$

where $\boldsymbol{\lambda}(k)$ is a costate variable. The Hamiltonian form of (1) and (2) can be written as the forward recursion[5]

$$\begin{bmatrix} \mathbf{x}(k+1) \\ \boldsymbol{\lambda}(k+1) \end{bmatrix} = \mathbf{H}\begin{bmatrix} \mathbf{x}(k) \\ \boldsymbol{\lambda}(k) \end{bmatrix} \tag{5}$$

where

$$\mathbf{H} = \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{-\mathrm{T}}\mathbf{Q} & -\mathbf{B}\mathbf{R}^{-1}\mathbf{A}^{-\mathrm{T}} \\ -\mathbf{A}^{-\mathrm{T}}\mathbf{Q} & \mathbf{A}^{-\mathrm{T}} \end{bmatrix}$$

with boundary conditions expressed in the standard form as

$$\mathbf{M}\begin{bmatrix} \mathbf{x}(0) \\ \boldsymbol{\lambda}(0) \end{bmatrix} + \mathbf{N}\begin{bmatrix} \mathbf{x}(n) \\ \boldsymbol{\lambda}(n) \end{bmatrix} = \mathbf{c} \tag{6}$$

Here $\mathbf{H}$ is the symplectic matrix, which has the property that its eigenvalues can be grouped into two disjoint subsets $\boldsymbol{\Gamma}_1$ and $\boldsymbol{\Gamma}_2$ such that for every $\boldsymbol{\lambda}_c \in \boldsymbol{\Gamma}_1$ there exists a $\boldsymbol{\lambda}_d \in \boldsymbol{\Gamma}_2$ which satisfies $\boldsymbol{\lambda}_c \times \boldsymbol{\lambda}_d = 1$ and we can choose either $\boldsymbol{\Gamma}_1$ or $\boldsymbol{\Gamma}_2$ to contain only the stable eigenvalues.[6]

Note that

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{F} & \mathbf{I} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{0} \end{bmatrix} \tag{7}$$

for the free-end optimal control problem and

$$\mathbf{M} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{x}(T) \end{bmatrix} \tag{8}$$

for the fixed-end optimal control problem.

Matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{S}$ have the forms

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \varepsilon\mathbf{A}_2 \\ \varepsilon\mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 & \varepsilon\mathbf{B}_2 \\ \varepsilon\mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}, \quad \mathbf{S} = \mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}} = \begin{bmatrix} \mathbf{S}_1 & \varepsilon\mathbf{Z} \\ \varepsilon\mathbf{Z}^{\mathrm{T}} & \mathbf{S}_2 \end{bmatrix} \tag{9}$$

The optimal open-loop control problem is a two-point boundary value problem with the associated state–costate equations forming the Hamiltonian matrix. For weakly coupled systems, after modifying some costate variables, the Hamiltonian matrix retains the weakly coupled form by interchanging some state and costate variables so that it can be block diagonalized via the non-singular transformation.[7] The idea of this paper is to exploit the reduced-order subsystems to find the optimal open-loop control in the new co-ordinates.

2. RECURSIVE REDUCED-ORDER SOLUTION OF THE OPEN-LOOP OPTIMAL CONTROL PROBLEM

Partitioning the vector $\boldsymbol{\lambda}(k)$ in (5) as $\boldsymbol{\lambda}(k)=[\boldsymbol{\lambda}_1^{\mathrm{T}}(k)\ \ \boldsymbol{\lambda}_2^{\mathrm{T}}(k)]^{\mathrm{T}}$ with $\boldsymbol{\lambda}_1(k)\in\mathbb{R}^{n_1}$ and $\boldsymbol{\lambda}_2(k)\in\mathbb{R}^{n_2}$, we get[8]

$$\begin{bmatrix}\mathbf{x}_1(k+1)\\ \mathbf{x}_1(k+1)\\ \boldsymbol{\lambda}_1(k+1)\\ \boldsymbol{\lambda}_1(k+1)\end{bmatrix}=\begin{bmatrix}\bar{\mathbf{A}}_1 & \varepsilon\bar{\mathbf{A}}_2 & \bar{\mathbf{S}}_1 & \varepsilon\bar{\mathbf{S}}_2\\ \varepsilon\bar{\mathbf{A}}_3 & \bar{\mathbf{A}}_4 & \varepsilon\bar{\mathbf{S}}_3 & \bar{\mathbf{S}}_4\\ \bar{\mathbf{Q}}_1 & \varepsilon\bar{\mathbf{Q}}_2 & \bar{\mathbf{A}}_{11}^{\mathrm{T}} & \varepsilon\bar{\mathbf{A}}_{21}^{\mathrm{T}}\\ \varepsilon\bar{\mathbf{Q}}_3 & \varepsilon\bar{\mathbf{Q}}_4 & \varepsilon\bar{\mathbf{A}}_{12}^{\mathrm{T}} & \bar{\mathbf{A}}_{22}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(k)\\ \mathbf{x}_2(k)\\ \boldsymbol{\lambda}_1(k)\\ \boldsymbol{\lambda}_2(k)\end{bmatrix}=\mathbf{H}\begin{bmatrix}\mathbf{x}_1(k)\\ \mathbf{x}_2(k)\\ \boldsymbol{\lambda}_1(k)\\ \boldsymbol{\lambda}_2(k)\end{bmatrix}\tag{10}$$

Interchanging the second and third rows in (10) produces

$$\begin{bmatrix}\mathbf{x}_1(k+1)\\ \boldsymbol{\lambda}_1(k+1)\\ \mathbf{x}_2(k+1)\\ \boldsymbol{\lambda}_2(k+1)\end{bmatrix}=\begin{bmatrix}\bar{\mathbf{A}}_1 & \bar{\mathbf{S}}_1 & \varepsilon\bar{\mathbf{A}}_2 & \varepsilon\bar{\mathbf{S}}_2\\ \bar{\mathbf{Q}}_1 & \bar{\mathbf{A}}_{11}^{\mathrm{T}} & \varepsilon\bar{\mathbf{Q}}_2 & \varepsilon\bar{\mathbf{A}}_{21}^{\mathrm{T}}\\ \varepsilon\bar{\mathbf{A}}_3 & \varepsilon\bar{\mathbf{S}}_3 & \bar{\mathbf{A}}_4 & \bar{\mathbf{S}}_4\\ \varepsilon\bar{\mathbf{Q}}_3 & \varepsilon\bar{\mathbf{A}}_{12}^{\mathrm{T}} & \bar{\mathbf{Q}}_4 & \bar{\mathbf{A}}_{22}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\mathbf{x}_1(k)\\ \boldsymbol{\lambda}_1(k)\\ \mathbf{x}_2(k)\\ \boldsymbol{\lambda}_2(k)\end{bmatrix}=\mathbf{H}\begin{bmatrix}\mathbf{x}_1(k)\\ \boldsymbol{\lambda}_1(k)\\ \mathbf{x}_2(k)\\ \boldsymbol{\lambda}_2(k)\end{bmatrix}\tag{11}$$

where

$$\mathbf{T}_1=\begin{bmatrix}\bar{\mathbf{A}}_1 & \bar{\mathbf{S}}_1\\ \bar{\mathbf{Q}}_1 & \bar{\mathbf{A}}_{11}^{\mathrm{T}}\end{bmatrix},\quad \mathbf{T}_2=\begin{bmatrix}\bar{\mathbf{A}}_2 & \bar{\mathbf{S}}_2\\ \bar{\mathbf{Q}}_2 & \bar{\mathbf{A}}_{21}^{\mathrm{T}}\end{bmatrix},\quad \mathbf{T}_3=\begin{bmatrix}\bar{\mathbf{A}}_3 & \bar{\mathbf{S}}_3\\ \bar{\mathbf{Q}}_3 & \bar{\mathbf{A}}_{13}^{\mathrm{T}}\end{bmatrix},\quad \mathbf{T}_4=\begin{bmatrix}\bar{\mathbf{A}}_4 & \bar{\mathbf{S}}_4\\ \bar{\mathbf{Q}}_4 & \bar{\mathbf{A}}_{12}^{\mathrm{T}}\end{bmatrix}\tag{12}$$

Introducing the notation

$$\mathbf{U}(k)=\begin{bmatrix}\mathbf{x}_1(k)\\ \boldsymbol{\lambda}_1(k)\end{bmatrix},\qquad \mathbf{V}(k)=\begin{bmatrix}\mathbf{x}_2(k)\\ \boldsymbol{\lambda}_2(k)\end{bmatrix}\tag{13}$$

we get the weakly coupled discrete system

$$\mathbf{U}(k+1)=\mathbf{T}_1\mathbf{U}(k)+\varepsilon\mathbf{T}_2\mathbf{V}(k),\qquad \mathbf{V}(k+1)=\varepsilon\mathbf{T}_3\mathbf{U}(k)+\mathbf{T}_4\mathbf{V}(k)\tag{14}$$

Applying the non-singular transformation[7]

$$\mathbf{P}=\begin{bmatrix}\mathbf{I} & -\varepsilon\mathbf{L}\\ \varepsilon\mathbf{K} & \mathbf{I}-\varepsilon^2\mathbf{KL}\end{bmatrix},\qquad \mathbf{P}^{-1}=\begin{bmatrix}\mathbf{I}-\varepsilon^2\mathbf{LK} & \varepsilon\mathbf{L}\\ -\varepsilon\mathbf{K} & \mathbf{I}\end{bmatrix}\tag{15}$$

to (14), i.e. $[\mathbf{U}^{\mathrm{T}}(k)\ \ \mathbf{V}^{\mathrm{T}}(k)]^{\mathrm{T}}=\mathbf{P}\times[\bar{\mathbf{U}}^{\mathrm{T}}(k)\ \ \bar{\mathbf{V}}^{\mathrm{T}}(k)]^{\mathrm{T}}$, produces two completely decoupled subsystems

$$\bar{\mathbf{U}}(k+1)=(\mathbf{T}_1-\varepsilon^2\mathbf{LT}_3)^k\bar{\mathbf{U}}(k),\qquad \bar{\mathbf{V}}(k+1)=(\mathbf{T}_4+\varepsilon^2\mathbf{T}_3\mathbf{L})^k\bar{\mathbf{V}}(k)\tag{16}$$

where $\mathbf{L}$ and $\mathbf{K}$ satisfy

$$\mathbf{K}(\mathbf{T}_1-\varepsilon^2\mathbf{LT}_3)-(\mathbf{T}_4+\varepsilon^2\mathbf{T}_3\mathbf{L})\mathbf{K}+\mathbf{T}_3=\mathbf{0},\qquad \mathbf{T}_1\mathbf{L}+\mathbf{T}_2-\mathbf{LT}_4-\varepsilon^2\mathbf{LT}_3\mathbf{L}=\mathbf{0}\tag{17}$$

Using numerical techniques,[7] matrices $\mathbf{L}$ and $\mathbf{K}$ can be obtained by solving (17) with the required accuracy, where the rate of convergence is $O(\varepsilon^{2j})$. Thus after j iterations one gets the approximations $\mathbf{L}^{(j)} = \mathbf{L} + O(\varepsilon^{2j})$ and $\mathbf{K}^{(j)} = \mathbf{K} + O(\varepsilon^{2j})$. Using $\mathbf{L}^{(j)}$ and $\mathbf{K}^{(j)}$ instead of $\mathbf{L}$ and $\mathbf{K}$ in (15) will perturb the coefficients (accuracy) of the corresponding system of linear difference equations (16) by $O(\varepsilon^{2j})$, which implies that the same accuracy of the system solutions is obtained.

The boundary conditions are changed owing to the interchange of $\boldsymbol{\lambda}_1(k)$ and $\mathbf{x}_2(k)$, which modifies the matrices in (6) as

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{U}(0) \\ \mathbf{V}(0) \end{bmatrix} + \mathbf{N}_1 \begin{bmatrix} \mathbf{U}(n) \\ \mathbf{V}(n) \end{bmatrix} = \mathbf{c}_1 \tag{18}$$

where

$$\mathbf{M}_1 = \begin{bmatrix} \mathbf{I}_{n_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{F}_1 & \mathbf{I}_{n_1} & -\varepsilon\mathbf{F}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\varepsilon\mathbf{F}_2^{\mathrm{T}} & \mathbf{0} & -\mathbf{F}_3 & \mathbf{I}_n \end{bmatrix}, \quad \mathbf{c}_1 = \begin{bmatrix} \mathbf{x}_1(0) \\ \mathbf{0} \\ \mathbf{x}_2(0) \\ \mathbf{0} \end{bmatrix} \tag{19}$$

Applying the non-singular transformation (15) to (18) produces

$$\mathbf{M}_2 \begin{bmatrix} \bar{\mathbf{U}}(0) \\ \bar{\mathbf{V}}(0) \end{bmatrix} + \mathbf{N}_2 \begin{bmatrix} \bar{\mathbf{U}}(n) \\ \bar{\mathbf{V}}(n) \end{bmatrix} = \mathbf{c}_1 \tag{20}$$

where

$$\mathbf{M}_2 = \mathbf{M}_1\mathbf{P}^{-1}, \qquad \mathbf{N}_2 = \mathbf{N}_1\mathbf{P}^{-1} \tag{21}$$

Solutions of (16) are then given by

$$\bar{\mathbf{U}}(k) = (\mathbf{T}_1 - \varepsilon^2\mathbf{L}\mathbf{T}_3)^k\bar{\mathbf{U}}(0), \qquad \bar{\mathbf{V}}(k) = (\mathbf{T}_4 + \varepsilon^2\mathbf{T}_3\mathbf{L})^k\bar{\mathbf{V}}(0) \tag{22}$$

We can eliminate $\bar{\mathbf{U}}(n)$ and $\bar{\mathbf{V}}(n)$ from (20), which leads to

$$\boldsymbol{\alpha}(\varepsilon) = \begin{bmatrix} \bar{\mathbf{U}}(0) \\ \bar{\mathbf{V}}(0) \end{bmatrix} = \mathbf{c}_1 \tag{23}$$

where

$$\boldsymbol{\alpha}(\varepsilon) = \mathbf{M}_2 + \mathbf{N}_2 \begin{bmatrix} (\mathbf{T}_1 - \varepsilon^2\mathbf{L}\mathbf{T}_3)^n & \mathbf{0} \\ \mathbf{0} & (\mathbf{T}_4 + \varepsilon^2\mathbf{T}_3\mathbf{L})^n \end{bmatrix} \tag{24}$$

It is shown in the Appendix that $\boldsymbol{\alpha}(\varepsilon)$ is non-singular, i.e. the decoupled system has a unique solution, assuming that the coupling parameter ε is sufficiently small. Since $\boldsymbol{\alpha}(\varepsilon)$ is invertible, $\bar{\mathbf{U}}(0)$ and $\bar{\mathbf{V}}(0)$ can be obtained.

Now we are able to find $\bar{\mathbf{U}}(k)$ and $\bar{\mathbf{V}}(k)$ from (22). Using (15), we can obtain $\mathbf{U}(k)$ and $\mathbf{V}(k)$. After getting the solutions of $\mathbf{U}(k)$ and $\mathbf{V}(k)$, we can use the following relations to obtain the values for $\boldsymbol{\lambda}_1(k)$ and $\boldsymbol{\lambda}_2(k)$:

$$\begin{bmatrix} \mathbf{x}_1(k) \\ \boldsymbol{\lambda}_1(k) \end{bmatrix} = \begin{bmatrix} \mathbf{U}_1(k) \\ \mathbf{U}_2(k) \end{bmatrix} = \mathbf{U}(k), \qquad \begin{bmatrix} \mathbf{x}_2(k) \\ \boldsymbol{\lambda}_2(k) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_1(k) \\ \mathbf{V}_2(k) \end{bmatrix} = \mathbf{V}(k), \qquad \boldsymbol{\lambda}(k) = [\boldsymbol{\lambda}_1^{\mathrm{T}}(k)\ \boldsymbol{\lambda}_2^{\mathrm{T}}(k)]^{\mathrm{T}} \tag{25}$$

The optimal control law is then obtained as

$$\mathbf{u}(k) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}(k+1) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1) + O(\varepsilon^{2j}) \tag{26}$$

where $\boldsymbol{\lambda}^{(j)}(k)$ is the costate variable corresponding to $\mathbf{L}^{(j)}$ and $\mathbf{K}^{(j)}$. Apparently, as j increases, the approximate control (26) converges very rapidly to the optimal solution. Kokotovic *et al.*[1] have shown how the control law (26) relates to ε ($\varepsilon = 0$ and $\varepsilon \neq 0$).

3. NUMERICAL EXAMPLE

In order to demonstrate the proposed method, a discrete system[9] is studied. The system matrices are

$$\mathbf{A} = \begin{bmatrix} 0\cdot964 & 0\cdot18 & 0\cdot017 & 0\cdot019 \\ -0\cdot342 & 0\cdot802 & 0\cdot162 & 0\cdot179 \\ 0\cdot016 & 0\cdot019 & 0\cdot983 & 0\cdot181 \\ 0\cdot144 & 0\cdot179 & -0\cdot163 & 0\cdot82 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0\cdot019 & 0\cdot001 \\ 0\cdot180 & 0\cdot019 \\ 0\cdot005 & 0\cdot019 \\ -0\cdot054 & 0\cdot181 \end{bmatrix}$$

with boundary condition and final time

$$\mathbf{x}^{\mathrm{T}}(0) = [1 \;\; 1 \;\; 1 \;\; 1], \qquad n = 5$$

The remaining matrices are chosen as $\mathbf{R} = I_2$, $\mathbf{Q} = 0\cdot1\mathbf{I}_4$ and the terminal condition $\mathbf{F} = 0\cdot5\mathbf{I}_4$. The small coupling parameter ε equals 0·329, which is estimated by the formula[10]

$$\varepsilon = \frac{\max(\|\mathbf{A}_2\|, \|\mathbf{A}_3\|)}{\max(\|\mathbf{A}_1\|, \|\mathbf{A}_4\|)} = \frac{0\cdot323}{0\cdot983} \approx 0\cdot329 \tag{27}$$

With the proposed method, simulation results for the open-loop optimal control (4) are obtained using the package MATLAB.[11] Table I gives the approximate and optimal values of the control $\mathbf{u}(k)$. The approximate control is defined as

$$\mathbf{u}^{(j)}(k) = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\lambda}^{(j)}(k+1) \tag{28}$$

where j stands for the number of iterations used to solve $\mathbf{L}$ and $\mathbf{K}$ in equation (17).

Table I. Approximate and optimal values of $\mathbf{u}(k)$

k	Approx. $\mathbf{u}(k)$ $j=0$	Approx. $\mathbf{u}(k)$ $j=1$	Approx. $\mathbf{u}(k)$ $j=2$	Approx. $\mathbf{u}(k)$ $j=3$	Approx. $\mathbf{u}(k)$ $j=4$	Optimal $\mathbf{u}(k)$ $j=5$
1	0·9806	0·3685	0·6464	0·4194	0·3999	0·3999
	−1·895	−0·2286	−0·3050	−0·2785	−0·2776	−0·2777
2	0·7421	0·3202	0·5465	0·3577	0·3405	0·3404
	−1·3454	−0·1304	−0·2526	−0·2314	−0·2303	−0·2303
3	0·5207	0·2742	0·4520	0·2966	0·2816	0·2815
	−0·8222	−0·0362	−0·2052	−0·1870	−0·1856	−0·1856
4	0·3157	0·2309	0·3634	0·2373	0·2243	0·2242
	−0·3382	0·0509	−0·1635	−0·1464	−0·1447	−0·1447
5	0·1259	0·1906	0·2809	0·1806	0·1697	0·1696
	0·0955	0·1276	−0·1284	−0·1109	−0·1087	−0·1088

4. CONCLUSIONS

The optimal finite time open-loop discrete control problem of weakly coupled systems is solved in terms of reduced-order problems with any desired accuracy. The proposed method reduces considerably the size of required computations and introduces full parallelism in the problem under study.

APPENDIX

Let the transition matrices of (16) be denoted by $\boldsymbol{\phi}(k)$ and $\boldsymbol{\psi}(k)$ respectively and let us partition them as

$$\boldsymbol{\phi}(k) = \begin{bmatrix} \phi_1(k) & \phi_2(k) \\ \phi_3(k) & \phi_4(k) \end{bmatrix} = (\mathbf{T}_1 - \varepsilon^2 \mathbf{L}\mathbf{T}_3)^k, \qquad \begin{bmatrix} \psi_1(k) & \psi_2(k) \\ \psi_3(k) & \psi_4(k) \end{bmatrix} = (\mathbf{T}_4 + \varepsilon^2 \mathbf{T}_3 \mathbf{L})^k$$

From (24) we have

$$\boldsymbol{\alpha}(\varepsilon) = \mathbf{M}_2 + \mathbf{N}_2 \begin{bmatrix} \boldsymbol{\phi}(n) & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\psi}(n) \end{bmatrix}$$

Using expressions for $\mathbf{M}_2$ and $\mathbf{N}_2$ defined by (21), we obtain

$$\boldsymbol{\alpha}(\varepsilon) = \begin{bmatrix} \mathbf{I}_{n_1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \phi_3(n) - \mathbf{F}_1\phi_1(n) & \phi_4(n) - \mathbf{F}_1\phi_2(n) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \psi_3(n) - \mathbf{F}_3\psi_1(n) & \psi_4(n) - \mathbf{F}_3\psi_2(n) \end{bmatrix} + O(\varepsilon)$$

Since the matrices $\phi_4(n) - \mathbf{F}_1\phi_2(n)$ and $\psi_4(n) - \mathbf{F}_3\psi_2(n)$ are invertible,[3,8] the matrix $\boldsymbol{\alpha}(\varepsilon)$ is invertible for sufficiently small values of ε.

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