9 Borghesani, C., Chait, Y., and Yaniv, O., *The QFT Toolbox for MATLAB*, The Mathwork, Inc., Natick, MA, 1995.
10 Saff, E. B., and Snider, A. D., *Fundamentals of Complex Analysis*, Prentice-Hall, NJ, 1976.
11 Doyle, J. C., and Stein, G., "Multivariable Feedback Design: Concepts for Classical/Modern Synthesis," *IEEE Trans.*, Vol. AC-26, 1981, pp. 4–16.
12 Chen, M. J., and Desoer, C. A., "Necessary and Sufficient Condition for Robust Stability of Linear Distributed Feedback Systems," *Int. J. Robust and Nonlinear Control*, Vol. 35, 1982, pp. 255–267.
13 Chait, Y., and Yaniv, O., "Multi-Input/Single-Output Computer-Aided Control Design Using the Quantitative Feedback Theory," *Int. J. Robust and Nonlinear Control*, Vol. 3, 1993, pp. 47–54.
14 Saleh, S. J., Barmish, B. R., and Petersen, I. R., "Synthesis of Robust Controllers with Few Degrees of Freedom for Systems with Structured Real Parametric Uncertainty," *Procs. IFAC*, Vol. 1, 1993, pp. 19–22.
15 Thompson, D. F., "Nondifferential Optimization Methods for QFT Controller Synthesis," *Procs. ASME WAM—Advances in Robust and Nonlinear Control Systems*, DSC-Vol. 53, 1993, pp. 87–94.
16 Zhao, Y., and Jayasuriya, S., "Robust Stabilization of Linear Time Invariant Systems with Parametric Uncertainties," *Advances in Robust and Nonlinear Control Systems*, DSC-Vol. 53, ASME WAM, New Orleans, 1993, pp. 79–86.

Filtering for Linear Stochastic Systems With Small Measurement Noise

Z. Aganovic,[1] Z. Gajic,[1] and X. Shen[2]

In this paper we present a method which produces complete decomposition of the optimal global Kalman filter for linear stochastic systems with small measurement noise into exact pure-slow and pure-fast reduced-order optimal filters both driven by the system measurements. The method is based on the exact decomposition of the global small measurement noise algebraic Riccati equation into exact pure-slow and pure-fast algebraic Riccati equations. An example is included in order to demonstrate the proposed method.

1 Introduction

Several authors have studied the limiting properties of the optimal Kalman filter (Friedland, 1971; Doyle, 1978; Halevi, 1986). In several papers the perfect measurement noise (Moylan, 1974; Haas, 1984; Shaked, 1986; Soroka and Shaked, 1988) and singular measurement noise (Bernstein and Hyland, 1985; Haddad and Bernstein, 1987; Halevi, 1989) problems are considered. However, the filtering of linear stochastic systems with small measurement noise, which is an important problem for several engineering areas such as signal processing, communications, and control theory, has been studied, up to our best knowledge, only in (Sachs, 1980; 1981).

Our approach to the filtering problem with small measurement noise employs the singular perturbation technique (Kokotovic et al., 1986; Gajic and Shen, 1993; Gajic and Lim, 1994). The use of singular perturbation method for this problem is also demonstrated in (Sachs, 1980; 1981). We have obtained the exact expressions for the optimal filters which are of reduced-order and driven by the system measurements. In addition, the optimal filter gains are completely determined in terms of the exact pure-slow and pure-fast reduced-order algebraic Riccati equations. Thus, we get the complete reduction in both off-line and on-line computations, so that the optimal filtering can be completely done at the local levels.

Consider the linear continuous-time invariant stochastic system

$$\dot{x}_1 = A_1x_1 + A_2x_2 + G_1w$$
$$\dot{x}_2 = A_3x_1 + A_4x_2 + G_2w \quad (1)$$

with the corresponding measurements corrupted by small noise (the noise smallness is represented by a small noise intensity matrix)

$$y = C_2x_2 + v \quad (2)$$

where $x_1 \in R^n$ and $x_2 \in R^m$ are state vectors, $w \in R^r$ is zero-mean, stationary, white Gaussian noise stochastic process with intensity $W > 0$, and $y \in R^m$ are the system measurements. The measurements are corrupted by small zero-mean, stationary, Gaussian white noise $v \in R^m$ whose intensity matrix is assumed to be $\epsilon^2 V > 0$ with ϵ being a small positive parameter. No loss of generality is incurred in (2) provided that the measurement matrix has full rank m, (Jameson and O'Malley, 1975)—this fact is also demonstrated in Example and Appendix. Thus, the problem is studied under the following assumption.

Assumption 1. $\det C_2^{m\times m} \neq 0$.

In the following A_i, G_j, C_2, $i = 1, 2, 3, 4$; $j = 1, 2$, are constant matrices.

The optimal Kalman filter, corresponding to (1)–(2), driven by the innovation process is given by (Kwakernaak and Sivan, 1972)

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + A_2\hat{x}_2 + K_1\nu$$
$$\dot{\hat{x}}_2 = A_3\hat{x}_1 + A_4\hat{x}_2 + K_2\nu$$
$$\nu = y - C_2\hat{x}_2 \quad (3)$$

with the optimal filter gain obtained from

$$K = PC^TV^{-1} = \begin{bmatrix} P_2C_2^TV^{-1} \\ \frac{1}{\epsilon}P_3C_2^TV^{-1} \end{bmatrix} = \begin{bmatrix} K_1 \\ \frac{1}{\epsilon}K_2 \end{bmatrix},$$
$$C^T = \begin{bmatrix} 0 \\ C_2^T \end{bmatrix}^T \quad (4)$$

where P is the positive semidefinite stabilizing solution of the algebraic Riccati equation

$$AP + PA^T - \frac{1}{\epsilon^2}PSP + GWG^T = 0 \quad (5)$$

with

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix},$$
$$S = C^TV^{-1}C, \quad P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & \frac{1}{\epsilon}P_3 \end{bmatrix} \quad (6)$$

It can be easily seen, by observing the form of the algebraic Riccati equation (5), that the above problem is dual to the corresponding cheap control problem (Jameson and O'Malley, 1975; Francis, 1977; Kokotovic et al., 1986; Huey and Gajic, 1993) with ϵ^2V playing the role of ϵ^2R, where R is the input penalty matrix) so that the filtering problem with small measurement noise can be studied as a singularly perturbed system.

[1] Rutgers University, Department of Electrical and Computer Engineering, Piscataway, NJ 08855-0909.
[2] Department of Electrical and Computer Engineering, University of Waterloo, Ontario, Canada N2L 3G1.
Contributed by the Dynamic Systems and Control Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERING. Manuscript received by the DSCD April 9, 1994; revised manuscript received June 27, 1994. Associate Technical Editor: B. Siciliano.

The filtering problem of singularly perturbed systems has been studied in (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977; Khalil and Gajic, 1984; Gajic, 1986; Gajic and Lim, 1994). The results of (Haddad, 1976; Haddad and Kokotovic, 1977; Teneketzis and Sandell, 1977) produce only $O(\epsilon)$ accuracy, whereas, the results of (Khalil and Gajic, 1984; Gajic, 1986; Gajic and Lim, 1994) produce an arbitrary order of accuracy. In the small measurement noise problem we cannot be pleased with $O(\epsilon)$[3] accuracy. For the decomposition and approximation of the singularly perturbed Kalman filter (3) the Chang transformation (Chang, 1972) has been used in (Khalil and Gajic, 1984; Gajic, 1986)

$$\begin{bmatrix} \hat{\eta}_1 \\ \hat{\eta}_2 \end{bmatrix} = \begin{bmatrix} I - \epsilon HL & -\epsilon H \\ L & I \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \tag{7}$$

where L and H satisfy algebraic equations

$$\begin{aligned} A_4 L - A_3 - \epsilon L(A_1 - A_2 L) &= 0 \\ -HA_4 + A_2 - \epsilon HLA_2 + \epsilon(A_1 - A_2 L)H &= 0 \end{aligned} \tag{8}$$

The Chang transformation applied to (3) produces independent slow ($\hat{\eta}_1$) and fast ($\hat{\eta}_2$) filters driven by the innovation process

$$\begin{aligned} \dot{\hat{\eta}}_1 &= (A_1 - A_2 L)\hat{\eta}_1 + (K_1 - HK_2 - \epsilon HLK_1)v \\ \epsilon\dot{\hat{\eta}}_2 &= (A_4 + \epsilon LA_2)\hat{\eta}_2 + (K_2 + \epsilon LK_1)v \end{aligned} \tag{9}$$

In the new coordinates the innovation process is given by

$$v = y + C_2 L\hat{\eta}_1 - (C_2 + \epsilon C_2 LH)\hat{\eta}_2 \tag{10}$$

Equations (5) and (8) are solvable and produce the unique solutions under the following assumptions.

Assumption 2. The matrix A_4 is invertible and the matrices in the Riccati Eq. (8) satisfy the standard stabilizability-detectability conditions (Khalil and Gajic, 1984; Gajic, 1986).

2 Exact Filter Decomposition

In the decomposition procedure from the previous section the slow and fast filters (9) require additional communication channels necessary to form the innovation process (10). In addition, the filter gains K_1 and K_2 are given in terms of the solution of the global algebraic Riccati Eq. (5). Here, we propose a decomposition scheme following the results of (Gajic and Lim, 1994) such that the slow and fast filters are completely decoupled and both of them are driven by the system measurements and the corresponding filter coefficients are obtained from the reduced-order exact slow and fast algebraic Riccati equations.

The method is based on the pure-slow pure-fast decomposition technique for solving the cheap control algebraic Riccati equation of singularly perturbed systems, (Gajic and Shen, 1993) and the slow-fast decomposition technique of (Gajic and Lim, 1994) derived for general singularly perturbed systems. We give a brief summary and an additional interpretation of the results of (Gajic and Shen, 1993), which will be used in this paper.

Consider the linear-quadratic optimal cheap control problem of (1), that is (Jameson and O'Malley, 1975; Kokotovic et al., 1986; Huey and Gajic, 1993)

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_2 x_2 \\ \dot{x}_2 &= A_3 x_1 + A_4 x_2 + B_2 u \end{aligned}$$

[3] By definition, $O(\epsilon) < K\epsilon$, where K is a bounded constant.

$$J = \int_0^\infty \left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \epsilon^2 u^T R u \right] dt,$$

$$Q \geq 0, \quad R > 0 \tag{11}$$

where the control vector, $u \in R^m$, has to be chosen such that the performance criterion, J, is minimized. The very well-known solution to this problem is given by

$$u = -R^{-1}B^T P_r x = -F_1 x_1 - F_2 x_2 \tag{12}$$

where P_r is the positive semidefinite solution of the regulator algebraic Riccati equation

$$A^T P_r + P_r A + Q - \frac{1}{\epsilon^2} P_r Z P_r = 0 \tag{13}$$

with

$$Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \quad Z = BR^{-1}B^T,$$

$$B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad P_r = \begin{bmatrix} P_{1r} & \epsilon P_{2r} \\ \epsilon P_{2r}^T & \epsilon P_{3r} \end{bmatrix} \tag{14}$$

Note that B_2 is a square nonsingular matrix.

The optimal regulator gains F_1 and F_2 are given by

$$F_1 = R^{-1}B_2^T P_{2r}^T, \quad F_2 = R^{-1}B_2^T P_{3r} \tag{15}$$

The results of interest that we need, which can be deduced from (Gajic and Shen, 1993) are given in the form of the following lemma.

Lemma 1. Consider the optimal *closed-loop* linear system

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + A_2 x_2 \\ \epsilon\dot{x}_2 &= (A_3 - B_2 F_1)x_1 + (A_4 - B_2 F_2)x_2 \end{aligned} \tag{16}$$

then there exists a nonsingular transformation $\mathbf{T}_1$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \mathbf{T}_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{17}$$

such that

$$\begin{aligned} \dot{\xi}_1 &= (a_1 + a_2 P_{rs})\xi_1 \\ \epsilon\dot{\xi}_2 &= (b_1 + b_2 P_{rf})\xi_2 \end{aligned} \tag{18}$$

where P_{rs} and P_{rf} are the unique solutions of the exact pure-slow and pure-fast completely decoupled algebraic Riccati equations

$$\begin{aligned} 0 &= P_{rs}a_1 - a_4 P_{rs} - a_3 + P_{rs}a_2 P_{rs} \\ 0 &= P_{rf}b_1 - b_4 P_{rf} - b_3 + P_{rf}b_2 P_{rf} \end{aligned} \tag{19}$$

Matrices a_i, b_i, $i = 1, 2, 3, 4$, can be found in (Gajic and Shen, 1993). The nonsingular transformation $\mathbf{T}_1$ is given by

$$\mathbf{T}_1 = (\Pi_1 + \Pi_2 P_r) \tag{20}$$

Even more, the global solution P_r can be obtained from the solutions of the reduced-order exact pure-slow and pure-fast algebraic Riccati equations as

$$P_r = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix}\right)\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_{rs} & 0 \\ 0 & P_{rf} \end{bmatrix}\right)^{-1} \tag{21}$$

Known matrices Ω_i, $i = 1, 2, 3, 4$ and Π_1, Π_2 are given in terms of the solutions of the Chang decoupling equations, (Gajic and Shen, 1993).

The desired slow-fast decomposition of the Kalman filter (3) will be obtained by producing a dual lemma to Lemma 1. Consider the optimal *closed-loop* Kalman filter (3) driven by the system measurements, that is

$$\dot{\hat{x}}_1 = A_1\hat{x}_1 + (A_2 - K_1C_2)\hat{x}_2 + K_1y$$
$$\epsilon\dot{\hat{x}}_2 = A_3\hat{x}_1 + (A_4 - K_2C_2)\hat{x}_2 + K_2y \quad (22)$$

with the optimal filter gains K_1 and K_2 calculated from (4)–(6). By duality between the optimal filter and regulator, the filter Riccati Eq. (5) can be solved by using the same decomposition method for solving (13) with

$$A \to A^T, \quad Q \to GWG^T, \quad F^T = K$$
$$Z = BR^{-1}B^T \to S = C^TV^{-1}C$$
$$K_1 = F_1^T, \quad K_2 = F_2^T, \quad F = [F_1 \quad F_2] \quad (23)$$

Consider the state-costate equations of the following system

$$\dot{x} = A^Tx - \frac{1}{\epsilon^2}C^TV^{-1}Cp \quad (24)$$

$$\dot{p} = -GWG^Tx - Ap \quad (24)$$

Introducing the partitioning of the state-costate variables as $x = [x_1^T \quad x_2^T]^T$ and $p = [p_1^T \quad \epsilon p_2^T]^T$ we get the singularly perturbed system of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \end{bmatrix} = T_1\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} + T_2\begin{bmatrix} x_2 \\ p_2 \end{bmatrix}$$

$$\epsilon\begin{bmatrix} \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = T_3\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} + T_4\begin{bmatrix} x_2 \\ p_2 \end{bmatrix} \quad (25)$$

where the matrices T_1, T_2, T_3, T_4 are given by

$$T_1 = \begin{bmatrix} A_1^T & 0 \\ -G_1WG_1^T & -A_1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_3^T & 0 \\ -G_1WG_2^T & -\epsilon A_2 \end{bmatrix}$$

$$T_3 = \begin{bmatrix} \epsilon A_2^T & 0 \\ -G_2WG_1^T & -A_3 \end{bmatrix}, \quad T_4 = \begin{bmatrix} \epsilon A_4^T & -C_2^TV^{-1}C_2 \\ -G_2WG_2^T & -\epsilon A_4 \end{bmatrix}$$

$$(26)$$

For the singularly perturbed linear system (25) the slow-fast decomposition is achieved by using the Chang decoupling equations

$$T_4M - T_3 - \epsilon M(T_1 - T_2M) = 0$$
$$-N(T_4 + \epsilon MT_2) + T_2 + \epsilon(T_1 - T_2M)N = 0 \quad (27)$$

These equations can be efficiently solved by either using the Newton method or the fixed point iterations, (Gajic and Shen, 1993). Also Eq. (27) can be solved by using the eigenvector approach (see Medanic, 1982).

Applying the Chang decomposition transformation to (25), we get

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\zeta}_1 \end{bmatrix} = (T_1 - T_2M)\begin{bmatrix} \eta_1 \\ \zeta_1 \end{bmatrix}$$

$$\epsilon\begin{bmatrix} \dot{\eta}_2 \\ \dot{\zeta}_2 \end{bmatrix} = (T_4 + \epsilon MT_2)\begin{bmatrix} \eta_2 \\ \zeta_2 \end{bmatrix} \quad (28)$$

By using the permutation matrices (Gajic and Shen, 1993)

$$E_1 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon}I_m \end{bmatrix},$$

$$E_2 = \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix} \quad (29)$$

we can relate variables in new and old coordinates by

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \zeta_1 \\ \zeta_2 \end{bmatrix} = E_2^T\begin{bmatrix} I - \epsilon NM & -\epsilon N \\ M & I \end{bmatrix}E_1\begin{bmatrix} x \\ p \end{bmatrix} \quad (30)$$

where $p = Px$. Introducing the notation

$$\Pi = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} = E_2^T\begin{bmatrix} I - \epsilon NM & M \\ -\epsilon N & I \end{bmatrix}E_1 \quad (31)$$

we get

$$\begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = [\Pi_1 \quad \Pi_2]\begin{bmatrix} x \\ p \end{bmatrix} = (\Pi_1 + \Pi_2P)x = \mathbf{T}_2x \quad (32)$$

Then, the desired transformation is given by

$$\mathbf{T}_2 = (\Pi_1 + \Pi_2P) \quad (33)$$

This transformation block diagonalizes the closed-loop system matrix $(A - KC)^T$, that is

$$\mathbf{T}_2(A - KC)^T\mathbf{T}_2^{-1}$$

is block diagonal, which follows by duality, (see (17)–(18)).

The similarity transformation $\mathbf{T}_2$ applied to the filter variables as

$$\begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix} = \mathbf{T}_2^{-T}\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} \quad (34)$$

produces

$$\begin{bmatrix} \dot{\hat{\eta}}_s \\ \dot{\hat{\eta}}_f \end{bmatrix} = \mathbf{T}_2^{-T}\begin{bmatrix} A_1 & A_2 - K_1C_2 \\ \frac{A_3}{\epsilon} & \frac{A_4 - K_2C_2}{\epsilon} \end{bmatrix}\mathbf{T}_2^T\begin{bmatrix} \hat{\eta}_s \\ \hat{\eta}_f \end{bmatrix}$$

$$+ \mathbf{T}_2^{-T}\begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix}y \quad (35)$$

such that the complete *closed-loop* decomposition is achieved, that is

$$\dot{\hat{\eta}}_s = (a_1 + a_2P_s)^T\hat{\eta}_s + K_sy$$
$$\epsilon\dot{\hat{\eta}}_f = (b_1 + b_2P_f)^T\hat{\eta}_f + K_fy \quad (36)$$

Note that the transposes of the system matrices in (36) come from the fact that the derivations have been performed by using duality. The matrices in (36) are given by

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = (T_1 - T_2 M),$$

$$\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = (T_4 + \epsilon M T_2), \begin{bmatrix} K \\ \frac{K_f}{\epsilon} \end{bmatrix} = \mathbf{T}_2^{-\mathbf{T}} \begin{bmatrix} K_1 \\ \frac{K_2}{\epsilon} \end{bmatrix} \quad (37)$$

$$0 = P_s a_1 - a_4 P_s - a_3 + P_s a_2 P_s$$

$$0 = P_f b_1 - b_4 P_f - b_3 + P_f b_2 P_f \quad (38)$$

The Newton method for solving nonsymmetric Riccati equations (38) can be found in (Gajic and Shen, 1993; Huey and Gajic, 1993). The nonsymmetric algebraic Riccati equation has been studied in (Medanic, 1982). An algorithm for solving the general nonsymmetric algebraic Riccati equation is derived in (Avramovic et al., 1980).

It is important to point out that the matrix P in (33) can be obtained in terms of P_s and P_f by using (21) with

$$P_{rs} = P_s, \quad P_{rf} = P_f \quad (39)$$

and Ω_1, Ω_2, Ω_3, Ω_4 obtained from

$$\Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} = E_1^{-1} \begin{bmatrix} I & \epsilon \mathrm{N} \\ -M & \mathrm{I} - \epsilon \mathrm{MN} \end{bmatrix} E_2^{-T} \quad (40)$$

A lemma dual to Lemma 1 can be now formulated as.

Lemma 2. Given the *closed-loop* optimal Kalman filter (22) of a linear singularly perturbed system, then there exists a nonsingular transformation matrix (33), which completely decouples (22) into pure-slow and pure-fast reduced-order filters (36) both driven by the system measurements. Even more, the decoupling transformation (33) and the filter coefficients given in (36) can be obtained in terms of the exact pure-slow and pure-fast reduced-order completely decoupled Riccati equations (38).

Note that the actual procedure for computing the filter decomposition can be completely done at the local levels. In the proposed method we have complete separation for both off-line (coefficient calculation) and on-line (filtering process itself) computations. The procedure is summarized in the form of the following algorithm.

Algorithm:

1. Solve Chang decoupling Eq. (27).
2. Find coefficients a_i, b_i, $i = 1, 2, 3, 4$ by using (37).
3. Solve the reduced-order algebraic Riccati Eqs. (38).
4. Form the transformation (33) by using (21), (31), and (39)–(40).
5. Calculate the reduced-order filter gains by using (4) and (37), where the matrix P is calculated from (21), (38)–(40).

3 Example

In order to demonstrate the proposed method we solve the control system positioning problem from (Kwakernaak and Sivan, 1972). The problem matrices are given by

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -4.6 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad W = 10$$

We assume that the measurements of angular displacement, x_1, and the angular velocity, x_2, are given as a linear combination corrupted by white measurement noise, that is

$$y(t) = [1 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + v(t)$$

with the measurement noise intensity taken from (Kwakernaak and Sivan, 1972) as $V = \epsilon^2 \times 0.1$ with $\epsilon = 0.001$. In order to achieve the measurement matrix form considered in this paper we apply the transformation from Appendix with

$$z(t) = M_1 z(t), \quad M_1 = \begin{bmatrix} 1 & 0.5 \\ -1 & 0.5 \end{bmatrix}, \quad N_1 = 1$$

which leads to the following matrices in the new coordinates (see Appendix)

$$A = \begin{bmatrix} -2.8 & 1.4 \\ 3.6 & -1.8 \end{bmatrix}, \quad G = \begin{bmatrix} -0.05 \\ 0.1 \end{bmatrix}$$

The measurement equation is now given by

$$y(t) = [0 \quad 1] z(t) + v_{\text{new}}(t),$$

$$\text{int}\{v_{\text{new}}(t)\} = N_1 V N_1 = 0.1\epsilon^2$$

The following pure-slow and pure-fast filters are obtained

$$\dot{\hat{\eta}}_1(t) = -\hat{\eta}_1(t) + y(t)$$

$$\epsilon\dot{\hat{\eta}}_2(t) = -\hat{\eta}_2(t) + 0.9982 y(t)$$

We have run the same problem with different values for a small parameter with the following results

$$\dot{\hat{\eta}}_1(t) = -0.999\hat{\eta}_1(t) + 0.999 y(t)$$

$$\epsilon\dot{\hat{\eta}}_2(t) = -1.001\hat{\eta}_2(t) + 0.981 y(t), \quad \epsilon = 0.01$$

$$\dot{\hat{\eta}}_1(t) = -0.9116\hat{\eta}_1(t) + 0.934 y(t)$$

$$\epsilon\dot{\hat{\eta}}_2(t) = -1.0970\hat{\eta}_2(t) + 0.769 y(t), \quad \epsilon = 0.1$$

It is important to notice that the actual value of the small positive parameter ϵ is also dependent on the system noise input matrix G and the intensity matrix of the system noise. It seems from our experience that the best estimate for ϵ is the "signal to noise ratio," (Anderson and Moore, 1979), that is

$$\epsilon^2 = \frac{\|GWG^T\|}{\|V\|}$$

where $\|\cdot\|$ is any norm and W and V are constant matrices.

4 Conclusion

In this paper we have exactly solved the small measurement noise problem by using the singular perturbation methodology. Two completely independent slow and fast reduced-order filters are obtained. In addition to the practical importance of the solved problem, we hope that the obtained results will bring deeper understanding of the effect of small measurement noise in the Kalman filtering since the slow and fast phenomena are now completely and exactly separated. Also we believe that the limiting perfect measurement noise case can be studied by using the results of this paper as an approximation for $\epsilon = 0$. It is our opinion that the discrete-time version of this problem might be an interesting area for future research.

APPENDIX

The following result is known from (Wilkinson, 1965). Given any matrix C of dimension $m \times n$ such that

$$\text{rank } C^{m \times n} = m$$

then there exist two nonsingular matrices $N_1^{m\times m}$, $M_1^{n\times n}$ such that

$$N_1^{m\times m}C^{m\times n}M_1^{n\times n} = [0 \quad I_m]^{m\times n}$$

Given a linear stochastic system of the form

$$\dot{x}(t) = Ax(t) + Gw(t)$$

$$y(t) = Cx(t) + v(t)$$

with constant matrices $A^{n\times n}$, $G^{n\times m}$, $C^{m\times n}$, $\text{int}(w) = W^{m\times m}$, $\text{int}(v) = V^{m\times m}$. Then, the following transformation

$$x = M_1 z$$

maps the given system into

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{G}w(t)$$

$$y_{\text{new}}(t) = [0 \quad I_r]z(t) + v_{\text{new}}(t)$$

where

$$\mathbf{A} = M_1^{-1}AM_1, \quad \mathbf{G} = M_1^{-1}G,$$

$$y_{\text{new}}(t) = N_1 y(t),$$

$$v_{\text{new}}(t) = N_1 v(t) \Rightarrow \text{int}(v_{\text{new}}(t)\} = N_1 V N_1^T$$

References

Anderson, B., and J. Moore, 1979, *Optimal Filtering*, Prentice Hall, Englewood Cliffs, NJ.

Avramovic, B., P. Kokotovic, J. Winkelman, and J. Chow, 1980. Area Decomposition for Electro-Mechanical Models of Power Systems, *Automatica*, Vol. 16, pp. 637–648.

Bernstein, D., and D. Hyland, 1985, "The Optimal Projection Equations for Reduced-Noise State Estimation," *IEEE Trans. Automatic Control*, Vol. AC-30, pp. 583–585.

Chang, K., 1972, "Singular Perturbations of a General Boundary Value Problem," *SIAM J. Math. Anal.*, Vol. 3, pp. 502–526.

Doyle, J., 1978, "Guaranteed Margins for LQG Regulators," *IEEE Trans. Automatic Control*, Vol. AC-23, pp. 756–757.

Francis, B., 1979, "The Optimal Linear-Quadratic Time-Invariant Regulator with Cheap Control," *IEEE Trans. Automatic Control*, Vol. AC-24, pp. 616–621.

Gajic, Z., 1986, "Numerical Fixed Point Solution for Near-Optimum Regulators of Linear Quadratic Gaussian Control Problems for Singularly Perturbed Systems," *Int. J. Control*, Vol. 43, pp. 373–387.

Gajic, Z., and X. Shen, 1993, *Parallel Algorithm for Optimal Control of Large Scale Linear Systems*, Springer-Verlag, London.

Gajic, Z., and M. Lim, 1994, "A New Filtering Method for Linear Singularly Perturbed Systems," *IEEE Trans. Automatic Control*, Vol. AC-39, pp. 1952–1955.

Friedland, B., 1971, "Limiting Forms for Optimal Stochastic Linear Regulators," ASME JOURNAL OF DYNAMIC SYSTEMS, MEASUREMENTS AND CONTROL, Vol. 93, pp. 134–141.

Haddad, A., 1976, "Linear Filtering of Singularly Perturbed Systems," *IEEE Trans. Automatic Control*, Vol. AC-21, pp. 515–519.

Haddad, A., and P. Kokotovic, 1977, "Stochastic Control of Linear Singularly Perturbed Systems," *IEEE Trans. Automatic Control*, Vol. AC-22, pp. 815–821.

Haddad, W., and D. Bernstein, 1987, "The Optimal Projection Equations for Reduced-Noise State Estimation: The Singular Measurement Noise." *IEEE Trans. Automatic Control*, Vol. AC-32, pp. 1135–1143.

Halevi, Y., 1986, "Extended Limiting Forms of Optimum Observers and LQG Regulators," *Int. J. Control*, Vol. 43, pp. 193–212.

Halevi, Y., 1989, "The Optimal Reduced-Order Estimator for Systems with Singular Measurement Noise," *IEEE Trans. Automatic Control*, Vol. AC-34, pp. 777–781.

Hass, V., 1984, "Reduced Order State Estimation for Linear Systems with Exact Measurements," *Automatica*, Vol. 20, pp. 225–229.

Huey, M., and Z. Gajic, 1993, Recursive Approach to High Gain and Cheap Control Problems, *Proc. World Congress of IFAC*, Vol. 9, pp. 289–292, Sydney, Australia.

Jameson, A., and R. O'Malley, 1975, "Cheap Control of Time-Invariant Regulator," *Applied Math. and Optimization*, Vol. 1, pp. 337–354.

Khalil, H., and Z. Gajic, 1984, "Near-Optimum Regulators for Stochastic Linear Singularly Perturbed Systems," *IEEE Trans. Automatic Control*, Vol. AC-29, pp. 531–541.

Kokotovic, P., H. Khalil, and J. O'Reilly, 1986, *Singular Perturbation Methods in Control: Analysis and Design*, Academic Press, Orlando.

Kwakernaak, H., and R. Sivan, 1972, "The Maximally Achievable Accuracy of Linear Optimal Regulators, and Linear Optimal Filters," *IEEE Trans. Automatic Control*, Vol. AC-17, pp. 79–86.

Medanic, J., 1982, "Geometric Properties of Invariant Manifolds of the Riccati Equation," *IEEE Trans. Automatic Control*, Vol. AC-27, pp. 670–677.

Moylan, P., 1974, "A Note on Kalman-Bucy Filters With Zero Measurement Noise," *IEEE Trans. Automatic Control*, Vol. AC-19, 263–264.

Sachs, S., 1980, *Asymptotic Analysis of Linear Filtering Problems*, Ph.D. thesis, Case Western Reserve University.

Sachs, S., 1981, "Asymptotic Analysis of the Small Observation Noise Linear Filtering Problems," *Proc. American Control Conference*, Sharlottesville, WA-4A.

Shaked, U., 1986, "Solution in the z-Domain to the Discrete-Time Stationary Kalman Filtering Problem for Systems with Perfect Measurements," *IEEE Trans. Automatic Control*, Vol. AC-31, pp. 1156–1159.

Soroka, E., and U. Shaked, 1988, "The LQG Optimal Regulator Problem for Systems with Perfect Measurements: Explicit Solution, Properties and Application to Practical Designs," *IEEE Trans. Automatic Control*, Vol. AC-33, pp. 941–944.

Su, W., Z. Gajic, and X. Shen, 1992, "The Exact Slow-Fast Decomposition of the Algebraic Riccati Equation of Singularly Perturbed Systems," *IEEE Trans. Automatic Control*, Vol. AC-37, pp. 1456–1459.

Teneketzis, D., and N. Sandell, 1977, "Linear Regulator Design for Stochastic Systems by Multiple Time-Scale Method," *IEEE Trans. Automatic Control*, Vol. AC-22, pp. 615–621.

Wilkinson, J., 1965, *The Algebraic Eigenvalue Problem*, Clarendon Press, Oxford, 1965.

Analysis of Induction Plasma Deposition Dynamics for Control

K. S. Narendra,[1] C. W. Smith,[1] and M. A. Gevelber[1]

Plasma torches are used in a variety of different applications including forming corrosion and wear resistant coatings, near net shape manufacture, and production of metal matrix composites. The dynamics of a low order nonlinear dynamic model of the process are analyzed to obtain insight into developing an appropriate control structure.

1 Introduction

Plasma deposition is a process in which metallic or ceramic particles are melted and accelerated in a plasma stream and subsequently made to impinge on a substrate to form a dense coating. The plasma is created by heating inert gases which, in the present case, is done using an induction torch. While significant research has been reported on the detailed modelling of the plasma deposition process (Boulos and Fauchais, 1986; Wei et al., 1988), little has been reported on addressing process control issues.

The present paper presents our analysis of the dynamics of a low order model (LOM) which captures the essential dynamics of the process (Narendra et al., 1994). The LOM is useful in revealing the basic limitations posed by the process dynamics and input/output coupling. The complete process model consists of three submodels which represent (*a*) the torch, (*b*) the jet, and (*c*) the plasma-particle interactions. The torch model is developed by lumping distinct flow regions and performing mass, momentum and energy balances on these regions. The important phenomena such as the magnetic pinch effect and recirculation zones have been included. The jet model is based on the analytical solution to the velocity and temperature fields in an axisymmetric jet while the plasma-particle thermal/fluid model is based on the laminar flow of a gas over a spherical

[1] College of Engineering, Manufacturing Engineers, Process Control Laboratory, Boston University, 44 Cummington St., Boston, MA 02215.

Contributed by the Dynamic Systems and Control Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. Manuscript received by the DSCD January 1992; revised manuscript received June 10, 1994. Associate Technical Editor: R. Shoureshi.