Composite Control of Discrete Singularly Perturbed Systems with Stochastic Jump Parameters

by X. SHEN *and* V.-G. GOURISHANKAR

Department of Electrical Engineering, University of Alberta, Edmonton, Canada T6G 2G7

and Q. XIA *and* M. RAO

Department of Chemical Engineering, University of Alberta, Edmonton, Canada T6G 2G6

ABSTRACT: *In this paper, a singular perturbation approach is presented to study discrete systems with stochastic jump parameters. The feedback controller design is decomposed into the design of slow and fast controllers which are combined to form the composite control. The multirate control structure allows the designer to accommodate multiple information rates and to implement required control computations. Conditions for complete separation of slow and fast regulator designs are given. It is shown that the composite feedback control is* $O(\varepsilon)$ *close to the optimal one, which yields an* $O(\varepsilon^2)$ *approximation of optimal performance.*

I. Introduction

Systems with stochastic jump parameters have been an attraction in system theory literature (**1–7**). The representations have been used in various fields ranging from economic processes (**1**) to power systems (**3**) when the system is subject to abrupt phenomena such as component and interconnection failure. The basic approach to this problem consists of the so-called jump linear quadratic regulator. The explicit control law can be obtained by solving a coupled set of Riccati-type equations. However, for the system with multirate modes, the feedback controller design usually suffers from the high dimensionality and ill-conditioning resulting from the interaction of slow and fast dynamic phenomena. In a recent paper (**8**), the singular perturbation approach, which has two remedial features—dimensional reduction and stiffness relief (**9, 10**), was presented to design a near-optimal regulator for the continuous-time system with stochastic jump parameters. In this paper, the basic features of the singular perturbation approach to continuous-time problems is extended to discrete-time ones. It is shown that the stiffness properties are taken advantage of by decomposing the original ill-conditioned system into well defined slow and fast subsystems: the slow optimal control problem will be a continuous-time problem, while the fast optimal control problem will be a discrete-time problem (**11**). The multirate control structure allows the designer to accom-

0016–0032(93)E0003–N

modate multiple information rates and to implement required control computations (**12**, **13**). The slow continuous-time controller and fast discrete-time controller are combined to form the composite controller for the whole discrete system with stochastic jump parameters. The composite feedback control is $O(\varepsilon)$ close to the optimal one, which yields an $O(\varepsilon^2)$ approximation of optimal performance. The conditions for the existence of composite optimal controllers for discrete singularly perturbed system with jump parameters are given.

II. Problem Formulation

Consider a discrete singularly perturbed system with random parameters (**10**, **11**)

$$x_1(k+1) = (I_1 + \varepsilon(r(k))A_1(r(k)))x_1(k) + \varepsilon(r(k))A_2(r(k))x_2(k) + \varepsilon(r(k))B_1(r(k))u(k) \quad (1)$$

$$x_2(k+1) = A_3(r(k))x_1(k) + A_4(r(k))x_2(k) + B_2(r(k))u(k) \quad (2)$$

$$y(k) = C_1(r(k))x_1(k) + C_2(r(k))x_2(k) \quad (3)$$

$$x_1(0) = x_{10}, \quad x_2(0) = x_{20} \quad (4)$$

where $x_i \in \mathscr{R}^{n_i}$, $i = 1,2$, comprise slow and fast state vectors, respectively, control $u(k)$ is an m vector and output $y(k)$ is an l vector. Equations (1)–(4) are intended to serve as a model of a system subject to abrupt changes in its coefficience matrices; e.g. sudden changes in an economic process. $\varepsilon(r(k))$ is a small random positive parameter. We wish to model the parameter processes with a discrete-time, discrete-state Markov jump process, i.e. $A_i(r(k))$, $B_l(r(k))$ and $C_l(r(k))$, $i = 1,2,3,4$; $l = 1,2$ are random signals depending on the plant mode $r(k)$ which is a stochastic jump process in a finite valuation set $S = \{1,2,\ldots,N\}$, with transition probabilities p_{ij}

$$\text{prob}\,\{r(k+1) = j \mid r(k) = i\} = p_{ij} \quad (5)$$

$$\sum_{j=1}^{N} p_{ij} = 1, \quad 0 \leqslant p_{ij} \leqslant 1. \quad (6)$$

With (1)–(6), consider the performance criterion

$$J(u, x(0), r(0)) = E\left\{\sum_{k=0}^{\infty} [y^{\mathrm{T}}(k)y(k) + u^{\mathrm{T}}(k)R(r(k))u(k)]\right\} \quad (7)$$

where

$$R(r(k)) = R^{\mathrm{T}}(r(k)) > 0$$

and $E\{\cdot\}$ denotes mathematical expectation. The optimal control is given by (**1**)

$$u_{\mathrm{opt}}(k) = -\left[R_i + B_i^{\mathrm{T}}\sum_{j=1}^{N} K_j p_{ij} B_i\right]^{-1}\left[B_i^{\mathrm{T}}\sum_{j=1}^{N} K_j p_{ij} A_i\right]x(k) \quad \text{when } r(k) = i \quad (8)$$

with

$$x(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix}$$

$$A_i = \begin{pmatrix} I_1 + \varepsilon(i)A_1(i) & \varepsilon(i)A_2(i) \\ A_3(i) & A_4(i) \end{pmatrix}$$

$$B_i = \begin{pmatrix} \varepsilon(i)B_1(i) \\ B_2(i) \end{pmatrix}$$

$$C_i = (C_1(i) \quad C_2(i))$$

where

$$K_i = A_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} A_i + C_i^{\mathrm{T}} C_i - \left(B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} A_i \right)^{\mathrm{T}} \times \left(R_i + B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} B_i \right)^{-1} \left(B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} A_i \right). \quad (9)$$

The necessary and sufficient conditions for the existence of steady-state optimal controllers were given in (**6**).

The minimum expected cost is

$$J = x_0^{\mathrm{T}} K_i x_0 \quad \text{when } r(0) = i. \quad (10)$$

The main goal of this paper is to present a new procedure to design a near-optimal controller in terms of the reduced-order slow and fast subsystems (**14**). There are two important reasons for this study: (1) to avoid an ill-conditioned numerical problem arising from the interaction between slow and fast modes associated with Eq. (9); and (2) to reduce the size of required computations and generate the near optimal solution in parallel—in slow and fast time-scales, and speed up the optimization processes.

III. Slow–Fast Decomposition

System (1)–(4) is singular as a function of $\varepsilon(r(k))$. We can observe order reduction and separation of time-scales as $\varepsilon(r(k)) \to 0$.

3.1. *Slow subsystem*

To obtain the slow subsystem, we neglect the effect of the fast modes, i.e. let $x_2(k+1) = x_2(k)$ in Eq. (2).

$$\bar{x}_1(k+1) = [I_1 + \varepsilon(r(k))A_1(r(k))]\bar{x}_1(k) + \varepsilon(r(k))A_2(r(k))\bar{x}_2(k) + \varepsilon(r(k))B_1(r(k))\bar{u}(k) \quad (11)$$

$$\bar{x}_2(k) = A_3(r(k))\bar{x}_1(k) + A_4(r(k))\bar{x}_2(k) + B_2(r(k))\bar{u}(k) \tag{12}$$

$$\bar{y}(k) = C_1(r(k))\bar{x}_1(k) + C_2(r(k))\bar{x}_2(k) \quad \bar{x}_1(0) = x_{10} \tag{13}$$

where a bar indicates a discrete quasi-steady state (**9**). Assuming that $[I_2 - A_4(r(k))]^{-1}$ exists, where I_2 is the identity matrix of dimension $n_2 \times n_2$, we express $\bar{x}_2(k)$ as

$$\bar{x}_2(k) = [I_2 - A_4(r(k))]^{-1}[A_3(r(k))\bar{x}_1(k) + B_2(r(k))\bar{u}(k)] \tag{14}$$

and substitute it into (7a) to eliminate $\bar{x}_2(k)$, the slow subsystem of (1)–(4) is defined by

$$x_s(k+1) = [I_1 + \varepsilon(r(k))A_s(r(k))]x_s(k) + \varepsilon(r(k))B_s(r(k))u_s(k) \tag{15}$$

$$y_s(k) = C_s(r(k))x_s(k) + D_s(r(k))u_s(k) \tag{16}$$

where

$$A_s(r(k)) = A_1(r(k)) + A_2(r(k))[I_2 - A_4(r(k))]^{-1}A_3(r(k))$$

$$B_s(r(k)) = B_1(r(k)) + A_2(r(k))[I_2 - A_4(r(k))]^{-1}B_2(r(k))$$

$$C_s(r(k)) = C_1(r(k)) - C_2(r(k))[I_2 - A_4(r(k))]^{-1}A_3(r(k))$$

$$D_s(r(k)) = C_2(r(k))[I_2 - A_4(r(k))]^{-1}B_2(r(k)).$$

Hence $\bar{x}_1(k) = x_s(k)$, $\bar{x}_2(k)$, $\bar{u}(k) = u_s(k)$ and $\bar{y}(k) = y_s(k)$ are the slow components of the corresponding variables in system (1)–(4). The solutions of Eqs (15) and (16) is sought in the time scale $t = \varepsilon k$ and (15) is rewritten as (**11**)

$$x_s(\varepsilon k + \varepsilon) - x_s(\varepsilon k) = \varepsilon A_s(r(k))x_s(\varepsilon(r(k))k) + \varepsilon B_s u_s(k). \tag{17}$$

Dividing both sides by $\varepsilon(r(k))$ and taking limit $\varepsilon \to 0$ yield

$$\frac{\mathrm{d}x_s(t)}{\mathrm{d}t} = A_s(r(t))x_s(t) + B_s(r(t))u_s(t) \tag{18}$$

$$y_s(t) = C_s(r(t))x_s(t) + D_s(r(t))u_s(t). \tag{19}$$

3.2. *Fast subsystem*

The fast subsystem is derived by making assumptions that $\bar{x}_1(k) = x_s(k) =$ constant and $\bar{x}_2(k+1) = \bar{x}_2(k)$. From Eqs (2), (3), (12) and (13) we obtain

$$x_2(k+1) - \bar{x}_2(k+1) = A_4(r(k))[x_2(k) - \bar{x}_2(k)] + B_2(r(k))[u(k) - u_s(k)] \tag{20}$$

$$y(k) - \bar{y}(k) = C_2(r(k))[x_2(k) - \bar{x}_2(k)]. \tag{21}$$

Defining $x_f(k) = x_2(k) - \bar{x}_2(k)$, $u_f(k) = u(k) - u_s(k)$ and $y_f(k) = y(k) - y_s(k)$, the fast subsystem (1)–(4) can be expressed as

$$x_f(k+1) = A_4(r(k))x_f(k) + B_2(r(k))u_f(k) \tag{22}$$

$$y_f(k) = C_2(r(k))x_f(k) \quad x_f(0) = x_{20} - \bar{x}_2(0). \tag{23}$$

In this way, the original discrete singularly perturbed system with random parameter is reduced to two reduced-order subsystems. We define the slow and fast performance criteria as

$$J_s = E\left\{\int_0^{\infty} (y_s^T(t)y_s(t) + u_s^T(t)R(r(t))u_s(t))\,d\tau \mid x(t_0), r(t_0)\right\} \tag{24}$$

and

$$J_f = E\left\{\sum_0^{\infty} (y_f^T(k)y_f(k) + u_f^T(k)R(r(k))u_f(k))\,d\tau \mid x(0), r(0)\right\}. \tag{25}$$

Lemma 1

The solution of the jump linear quadratic problem for the slow subsystem (15), (16) and (24) is given as

$$u_s(t) = F_{si}x_s(t) \quad \text{when } r(t) = i \tag{26}$$

where

$$F_{si} = R_{si}^{-1}(D_{si}^T C_{si} + B_{si}^T K_{si}) \tag{27}$$

with $R_{si} = R_i + D_{si}^T D_{si}$. K_{si} are the solutions of the following set of N coupled Riccati equations

$$0 = K_{si}(A_{si} - B_{si}R_{si}^{-1}D_{si}^T C_{si}) + (A_{si} - B_{si}R_{si}^{-1}D_{si}^T C_{si})^T K_{si} - K_{si}B_{si}R_{si}^{-1}B_{si}^T K_{si} + \sum_{j=1}^{N} K_{sj}p_{ij} - C_{si}^T(I_1 - D_{si}R_{si}^{-1}D_{si}^T)C_{si} \quad i = 1, \ldots, N. \tag{28}$$

Proof: Rewriting (24) with (16), we obtain

$$J_s = E\left\{\frac{1}{2}\int_0^{t_f} (x_s^T C_s(r(t))^T C_s(r(t))x_s + 2x_s^T C_s(r(\tau))^T D_s(r(\tau))u_s + u_s^T R_s(r(\tau))u_s)\,d\tau\right\} \tag{29}$$

where

$$R_s(r(t)) = R_i + D_s^T(r(t))D_s(r(t)). \tag{30}$$

Defining the Hamiltonian matrix

$$H(t, x_s, u_s) = \tfrac{1}{2}[x_s^T C_s(r(t))^T C_s(r(t))x_s + 2x_s^T C_s(r(t))^T D_s(r(t))u_s + u_s^T R_s(r(t))u_s] + \lambda^T(A_s(r(t))x_s + B_s(r(t))u_s). \tag{31}$$

From the stochastic maximum principle (**2**), the optimal control law $u_s^*(t)$ must satisfy

$$E\{H(t, x_s, u_s^*) \mid t, x_s, r(t)\} \leqslant E\{H(t, x_s, u_s) \mid t, x_s, r(t)\}. \tag{32}$$

As a consequence, we get

$$D_s(r(t))^T C_s(r(t))x_s + R_s(r(t))u_s + B_s(r(t))^T E\{\lambda \mid t, x_s, r(t)\} = 0 \tag{33}$$

or

$$u_s = -R_s(r(t))^{-1}[B_s(r(t))^T E\{\lambda \mid t, x_s, r(t)\} + D_s(r(t))^T C_s(r(t))x_s] \tag{34}$$

and

$$\dot{\lambda} = -C_s(r(t))^T C_s(r(t))x_s + C_s(r(t))^T D_s(r(t))R_s(r(t))^{-1}[B_s(r(t))^T E\{\lambda \mid t, x_s, r(t)\} + D_s(r(t))^T C_s(r(t))x_s] - A_s(r(t))^T \lambda. \tag{35}$$

Let

$$\lambda = K_s x_s \tag{36}$$

we have

$$\begin{aligned}\dot{\lambda} &= \dot{K}_s x_s + K_s \dot{x}_s \\ &= \dot{K}_s x_s + K_s[A_s(r(t))x_s \\ &\quad - B_s(r(t))R_s(r(t))^{-1}D_s(r(t))^T C_s(r(t))x_s \\ &\quad - B_s(r(t))R_s(r(t))^{-1}E\{K_s \mid t, r(t)\}x_s].\end{aligned} \tag{37}$$

Comparing (35) and (37), we get

$$\begin{aligned}&\dot{K}_s x_s + K_s[A_s(r(t))x_s - B_s(r(t))R_s(r(t))^{-1}D_s(r(t))^T C_s(r(t))x_s \\ &\quad - B_s(r(t))R_s(r(t))^{-1}E\{K_s \mid t, r(t)\}x_s] = -C_s(r(t))^T C_s(r(t))x_s \\ &\quad + C_s(r(t))^T D_s(r(t))R_s(r(t))^{-1}[B_s(r(t))^T E\{K_s \mid t, r(t)\} \\ &\quad + D_s(r(t))^T C_s(r(t))]x_s - A_s(r(t))^T K_s x_s \\ &\qquad K_s(t_f) = 0.\end{aligned} \tag{38}$$

Since $K_s(t)$ is independent of $x_s(t)$ and is symmetric with the indicated boundary condition, from (38) we have

$$\begin{aligned}-\dot{K}_s &= A_s(r(t))^T K_s + K_s A_s(r(t)) \\ &\quad - C_s(r(t))^T D_s(r(t))R_s(r(t))^{-1}B_s(r(t))^T E\{K_s \mid t, r(t)\} \\ &\quad - K_s B_s(r(t))R_s(r(t))^{-1}D_s(r(t))^T C_s(r(t)) \\ &\quad - K_s B_s(r(t))R_s(r(t))^{-1}E\{K_s \mid t, r(t)\} + C_s(r(t))^T C_s(r(t))x_s \\ &\quad - C_s(r(t))^T D_s(r(t))R_s(r(t))^{-1}D_s(r(t))^T C_s(r(t)) \\ &\qquad K_s(t_f) = 0.\end{aligned} \tag{39}$$

Defining

$$[A_{si}, B_{si}, C_{si}, D_{si} = [A_s(r(t)), B_s(r(t)), C_s(r(t)), D_s(r(t)) \mid r(t) = i]$$

$$E\{K_s \mid t, r(t) = i\} = K_{si}$$

then

$$-E\{\dot{K}_s \mid t, r(t) = i\} = A_{si}^T K_{si} + K_{si} A_{si} - C_{si}^T D_{si} R_{si}^{-1} B_{si}^T K_{si}$$
$$- K_{si} B_{si} R_{si}^{-1} D_{si}^T C_{si} - K_{si} B_{si} R_{si}^{-1} K_{si} + C_{si}^T C_{si} - C_{si}^T D_{si} R_{si}^{-1} D_{si}^T C_{si}. \quad (40)$$

The quantity $E\{K_s(t) \mid t, r(t) = i\}$ is computed by

$$E\{\dot{K}_s(t) \mid r(t) = i\} \quad (41)$$

$$= \lim_{\Delta \to 0} \frac{E\{K_s(t+\Delta) \mid r(t) = i\} - E\{K_s(t) \mid r(t) = i\}}{\Delta}. \quad (42)$$

For the jump linear system described by (15) and (16), it is true that

$$E\{K_s(t+\Delta) \mid r(t) = i\} = K_{si}(t+\Delta) + \Delta \sum_{j=1}^{N} K_{sj}(t) p_{ij} + O(\Delta^2) \quad (43)$$

thus (41) yields

$$E\{\dot{K}_s \mid r(t)\} = \dot{K}_{si} + \sum_{j=1}^{N} K_{si} p_{ij}. \quad (44)$$

Substituting (43) into (40) and rearranging it, we obtained (28) for the steady-state case. From Eq. (34), it is clear that

$$u_s = -F_{si} x_s \quad \text{when} \quad r(t) = i \quad (45)$$

where F_{si} is as shown in Eq. (27). ■

Lemma 2

The solution of the jump linear quadratic problem for the fast subsystem (22), (23) and (25) is given as

$$u_f(k) = -F_{fi} x_f(k) \quad \text{when} \quad r(k) = i \quad (46)$$

where

$$F_{fi} = \left[R_i + B_{2i}^T \sum_{j=1}^{N} K_{fj} p_{ij} B_{2i} \right]^{-1} \left[B_{2i}^T \sum_{j=1}^{N} K_{fj} p_{ij} A_{4i} \right]. \quad (47)$$

K_{fi} is the solution of the following set of N coupled Riccati equations

$$K_{fi} = A_{4i}^T \sum_{j=1}^{N} K_{fj} p_{ij} A_{4i} + C_{2i}^T C_{2i} - \left(B_{2i}^T \sum_{j=1}^{N} K_{fj} p_{ij} A_{4i} \right)^T$$
$$\times \left(R_i + B_{2i}^T \sum_{j=1}^{N} K_{fj} p_{ij} B_{2i} \right)^{-1} \left(B_{2i}^T \sum_{j=1}^{N} K_{fj} p_{ij} A_{4i} \right) \quad i = 1, \ldots, N. \quad (48)$$

Proof: The optimal control law for the fast subsystem can be obtained by directly applying Eqs (8) and (9). ■

3.3. *Composite control*

With the solutions of the slow and fast problems in hand, the composite control law for the whole system (1)–(4) is then given by

$$\begin{aligned} u_c &= u_s(t) + u_f(k) \\ &= -F_{si}x_s(t) - F_{fi}x_f(k). \end{aligned} \tag{49}$$

With $x_2(k) - \bar{x}_2(t)$ replacing $x_f(k)$, expressing $\bar{x}_2(t)$ in terms of $x_s(t)$ and $u_s(t)$, and $x_1(k)$ replacing $x_s(t)$, we obtain

$$\begin{aligned} u_c &= -F_{si}x_1(k) - F_{fi}[x_2(k) - (I_2 A_{4i})^{-1}(A_{3i} - B_{2i}F_{si})x_1(k)] \\ &= -[F_{si} - F_{fi}(I_2 - A_{4i})^{-1}(A_{3i} - B_{2i}F_{si})]x_1(k) - F_{fi}x_2(k) \\ &\qquad \text{when } r(k) = i. \end{aligned} \tag{50}$$

Assumption 1. Transition of plant mode $r(k)$ will not change the fast- and slow-mode distributions of the system, that is, the following condition should hold

$$\text{Max}\,|\text{eig}\,\{A_4(r(k))\}| \ll \text{Min}\,|\text{eig}\,\{A_s(r(k))\}|.$$

Assumption 2. $r(k)$ has low switch rates. The triples $(A_s(r(t)), B_s(r(t)), C_s(r(t)))$ and $(A_4(r(k)), B_2(r(k)), C_2(r(k)))$ are stabilizable and detectable, respectively, in all plant modes $r(k) \in S$.

The following theorem gives the suboptimality of the composite control.

Theorem

Under the assumptions, the composite control (49) is $O(\varepsilon^2)$ near-optimal in the sense that

$$J - J_{\text{opt}} = O(\varepsilon^2) \tag{51}$$

where

$$\varepsilon = \text{Max}\,\{\varepsilon(r(k)) \mid r(k) = 1, \ldots, N\}.$$

Proof: We rewrite the optimal control law given in (8) as

$$u_{\text{opt}} = -(F^0_{1i}F^0_{2i})x(k) = -F^0_i x(k) \quad \text{when } r(k) = i \tag{52}$$

where $x(k) = (x_1^{\mathrm{T}}(k)x_2^{\mathrm{T}}(k))^{\mathrm{T}}$,

$$F^0_i = \left[R_i + B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} B_i\right]^{-1} \left[B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} A_i\right]$$

then

$$J_{\text{opt}} = x^{\mathrm{T}}(0)K_i x(0) \quad \text{when } r(0) = i \tag{53}$$

where K_i satisfies Eq. (9). Similarly, the composite feedback control can be represented as

$$u = -(F_{1i}F_{2i})x(k) = -F_i x(k) \quad \text{when } r(k) = i. \tag{54}$$

The closed-loop system under the composite feedback control (49) is

$$x(k+1) = (A_i - B_i F_i)x(k) \quad \text{when } r(k) = i \tag{55}$$

and the performance index is

$$J = x^{\mathrm{T}}(0)K'_i x(0) \quad \text{when } r(0) = i \tag{56}$$

where K'_i satisfies the N coupled Lyapunov equation

$$K'_i = (A_i - B_i F_i)^{\mathrm{T}} \bar{K}'_i (A_i - B_i F_i) + C_i^{\mathrm{T}} C_i + F_i^{\mathrm{T}} R_i F_i \quad \text{when } r(k) = i. \tag{57}$$

Let

$$\begin{aligned} \bar{K}'_i &= \sum_{j=1}^{N} K'_j p_{ij} \\ \bar{K}_i &= \sum_{j=1}^{N} K_j p_{ij} \\ \bar{R}_i &= R_i + B_i^{\mathrm{T}} \sum_{j=1}^{N} K_j p_{ij} B_i = R_i + B_i^{\mathrm{T}} \bar{K}_i B_i. \end{aligned} \tag{58}$$

Subtracting (9) from (57) yields

$$\begin{aligned} K'_i - K_i^0 &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) \\ &\quad + (A_i - B_i F_i)^{\mathrm{T}} \bar{K}_i (A_i - B_i F_i) - A_i^{\mathrm{T}} \bar{K}_i A_i \\ &\quad + F_i^{\mathrm{T}} R_i F_i + A_i^{\mathrm{T}} \bar{K}_i B_i \bar{R}_i^{-1} B_i^{\mathrm{T}} \bar{K}_i A_i \\ &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) - F_i^{\mathrm{T}} B_i^{\mathrm{T}} \bar{K}_i A_i \\ &\quad - A_i^{\mathrm{T}} \bar{K}_i B_i F_i + F_i^{\mathrm{T}} B_i^{\mathrm{T}} \bar{K}_i B_i F_i \\ &\quad + F_i^{\mathrm{T}} R_i F_i + A_i^{\mathrm{T}} \bar{K}_i B_i \bar{R}_i^{-1} B_i^{\mathrm{T}} \bar{K}_i A_i \\ &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) \\ &\quad - F_i^{\mathrm{T}} \bar{R}_i \bar{R}_i^{-1} B_i^{\mathrm{T}} \bar{K}_i A_i - A_i^{\mathrm{T}} \bar{K}_i B_i \bar{R}_i^{-1} \bar{R}_i F_i \\ &\quad + F_i^{\mathrm{T}} \bar{R}_i F_i + A_i^{\mathrm{T}} \bar{K}_i B_i \bar{R}_i^{-1} B_i^{\mathrm{T}} \bar{K}_i A_i \\ &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) \\ &\quad + [F_i^{\mathrm{T}} - A_i^{\mathrm{T}} \bar{K}_i B_i \bar{R}_i^{-1}] \bar{R}_i [F_i - \bar{R}^{-1} B_i^{\mathrm{T}} \bar{K}_i A_i] \\ &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) + [F_i - F_i^0]^{\mathrm{T}} \bar{R}_i [F_i - F_i^0] \\ &= (A_i - B_i F_i)^{\mathrm{T}} (\bar{K}'_i - \bar{K}_i)(A_i - B_i F_i) \\ &\quad + [F_i - F_i^0]^{\mathrm{T}} (R_i + B_i^{\mathrm{T}} \bar{K}_i B_i)[F_i - F_i^0]. \end{aligned} \tag{59}$$

It can be shown (**15**) that the term inside the last bracket is

$$[F_i - F_i^0] = [F_{1i} - F_{1i}^0, F_{2i} - F_{2i}^0] = O(\varepsilon). \tag{60}$$

Let $K'_i - K_i = V_i$ and $\bar{K}'_i - \bar{K}_i = \bar{V}_i$, (59) then becomes

$$V_i = (A_i - B_i F_i)^{\mathrm{T}} \bar{V}_i (A_i - B_i F_i) + O(\varepsilon^2). \tag{61}$$

By application of the implicit function theorem it can be shown (**15**) that K'_i and K_i possess a power series expansion at $\varepsilon = 0$. So V_i can be expanded as

$$V_i = \sum_{m=0}^{\infty} \frac{\varepsilon^m}{m!} \begin{pmatrix} V_{1i}^{(m)}/\varepsilon & V_{2i}^{(m)} \\ V_{2i}^{(m)\mathrm{T}} & V_{3i}^{(m)} \end{pmatrix} \tag{62}$$

Partitioning (61) and matching the zero order terms, it can be easily shown that $V_{3i}^{(0)} = 0$, $V_{1i}^{(0)}$ and $V_{2i}^{(0)} = 0$. Repeating the same argument, it can be shown that $V_{li}^{(1)} = 0$, $l = 1, 2, 3$. ■

IV. Conclusions

The near-optimal control of the singular perturbed systems has been studied when the system and cost parameters jump randomly according to a finite Markov process. The results obtained above suggest that the subcontrollers be separately designed to satisfy the corresponding performance index and implemented as the composite controller. The approach avoids the ill-conditioning of the original problem and reduces the size of required computations.

References

(1) W. P. Blair Jr and D. D. Sworder, "Feedback control of a class of a linear discrete systems with jump parameters and quadratic cost criteria", *Int. J. Control*, Vol. 21, pp. 833–841, 1975.

(2) D. D. Sworder and D. S. Chou, "A survey of some design methods for random parameter system", "Proc. 24th CDC", pp. 894–899, Ft. Lauderdale, FL, 1985.

(3) R. Malhame and C. Y. Chong, "Electric load model synthesis by diffusion approximation of a high-order hybrid-state stochastic system", *IEEE Trans. Autom. Control*, Vol. AC-30, pp. 854–860, 1985.

(4) M. Mariton, "On controllability of linear systems with stochastic jump parameters", *IEEE Trans. Autom. Control*, Vol. AC-31, pp. 680–683, 1986.

(5) M. D. Fragoso, "Discrete-time jump LQG problem", *Int. J. Systems Sci.*, Vol. 20, pp. 2539–2545, 1989.

(6) H. J. Chizeck, A. S. Willsky and D. Castanon, "Discrete-time Markovian-jump linear quadratic optimal control", *Int. J. Control*, Vol. 43, pp. 213–231, 1986.

(7) X. Feng, K. A. Loparo, Y. Ji and H. J. Chizeck, "Stochastic stability properties of jump linear systems", *IEEE Trans. Autom. Control*, Vol. AC-37, pp. 38–53, 1992.

(8) X. Shen, Q. Xia and M. Rao, "Near-optimal regulators for singularly perturbed system with random parameters", *Control-Theory Adv. Technol.*, Vol. 9, pp. 759–773, 1993.

(9) P. Kokotovic, H. Khalil and J. O'Reilly, "Singular Perturbation Methods in Control: Analysis and Design", Academic Press, New York, 1986.

(10) Z. Gajic, Dj. Petkovski and X. Shen, "Singularly Perturbed and Weakley Coupled Control Systems—A Recursive Approach", Springer, New York, 1990.

(11) B. Litkouhi and H. Khalil, "Infinite-time regulators for singularly perturbed difference equations", *Int. J. Control*, Vol. 39, pp. 587–598, 1984.

(12) G. Blankenship, "Singularly perturbed difference equations in optimal control problems", *IEEE Trans. Autom. Control*, Vol. AC-26, pp. 911–917, 1981.

(13) M. S. Mahmoud, "Structural properties of discrete systems with slow and fast modes", *Large Scale Systems*, Vol. 3, pp. 227–236, 1982.

(14) B. Litkouhi and H. Khalil, "Multirate and composite control of two-time-scale discrete-time systems", *IEEE Trans. Autom. Control*, Vol. AC-30, pp. 645–651, 1985.
(15) B. Litkouhi, "Sample-Data Control of Systems with Slow and Fast Modes", Ph.D. Dissertation, Michigan State University, 1983.

Received: 14 July 1993
Accepted: 30 September 1993