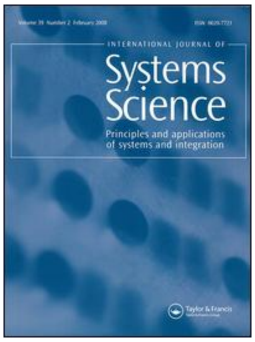

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INTERNATIONAL JOURNAL OF SYSTEMS SCIENCE

ISSN: 0020-7721 (Print) 1464-5319 (Online) Journal homepage: www.tandfonline.com/journals/tsys20

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INT. J. SYSTEMS SCI., 1994, VOL. 25, NO. 12, 2235–2244

Optimal control for large-scale systems: a recursive approach

X. SHEN†, Q. XIA‡, M. RAO‡ and V. GOURISHANKAR‡

A recursive fixed-point-type method is presented for the study of the optimal control problem of large-scale systems. The control is obtained by decomposition of the system to 'ϵ coupled' subsystems so that only low-order systems are involved in algebraic computations. It is shown that the developed reduced-order parallel algorithms converge to the desired solution with the rate $O(\epsilon)$. Owing to its recursive nature, the presented method produces a considerable saving of computation. An illustrative numerical example is given to verify the proposed approach.

1. Introduction

The numerical calculation of the exact optimal control is impracticable owing to the complexity of large-scale systems. To reduce the computations and investigate the properties of large-scale systems, intuition and experience may indicate how to split a large-scale design problem into a set of simpler subsystem problems. Examples of this approach in process control, power system, space guidance and other fields of control engineering are numerous. Recently, the problem of decomposition of a large-scale system based on the perturbation method has received the attention of many authors (Kokotovic *et al.* 1969, Delacour *et al.* 1978, Sandell *et al.* 1978, Jamshidi 1983). The control is obtained by decomposition of the system to what are called 'ϵ-coupled' subsystems, i.e. the feedback gain matrix of the system is defined as a power series of the coupling parameter ϵ. The retaining n terms in the expansion give an approximation of order $2n$ to the optimal cost. The zeroth-order terms are computed from decoupled Riccati equations obtained by setting $\epsilon = 0$ in the system matrices, and the higher-order terms are computed by solving decoupled linear equations. Because it is non-recursive in nature, the power-series expansion method becomes very cumbersome and computationally very expensive when a high order of accuracy is required, and thus difficult, to implement.

In this paper a recursive fixed point type algorithm is presented to obtain the optimal control in terms of 'ϵ-coupled' subsystems. This algorithm was first developed by Shen and Gajic (1990) and Gajic *et al.* (1990) for two subsystems only. It has been shown that the recursive method is particularly useful when the coupling parameter ϵ is not small and/or when some desired order of accuracy is required, namely $O(\epsilon^{2k})$, where $k = 1, 2, 3, 4, \ldots$ We will extend the approach to N subsystems. It is proved that the convergence rate of the algorithm is $O(\epsilon^2)$ for subsystem problems, and $O(\epsilon)$ for co-ordinator's problems. Owing to its recursive nature, this method is conceptually simple and very suitable for parallel programming. An illustrative numerical example is given to verify the proposed approach.

Received 15 February 1993.
† Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1.
‡ Department of Electrical Engineering, University of Alberta, Edmonton, Canada T6G 2G7.

2. Problem formulation

The linear quadratic optimal control problem requires one to find a control u which minimizes a chosen cost function of the form

$$J = \frac{1}{2}\int_0^\infty [x^T \mathbf{Q} x + u^T \mathbf{R} u]\, dt \tag{1}$$

subject to the dynamics of the system

$$\dot{x} = \mathbf{A}x + \mathbf{B}u \tag{2}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is the control input. The system is decomposed to N 'ϵ-coupled' subsystems so that the system matrices $\mathbf{A}$ and $\mathbf{B}$ have the following form (Delacour *et al.* 1978, Nath *et al.* 1985).

$$\mathbf{A} = \begin{pmatrix} A_{11} & \epsilon A_{12} & \cdots & \epsilon A_{1N} \\ \epsilon A_{21} & A_{22} & \cdots & \epsilon A_{2N} \\ \vdots & & \ddots & \vdots \\ \epsilon A_{N1} & \cdots & \epsilon A_{N(N-1)} & A_{NN} \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} B_{11} & \epsilon B_{12} & \cdots & \epsilon B_{1M} \\ \epsilon B_{21} & B_{22} & \cdots & \epsilon B_{2M} \\ \vdots & & \ddots & \vdots \\ \epsilon B_{N1} & \cdots & \epsilon B_{N(M-1)} & B_{NM} \end{pmatrix}$$

where ϵ is a small coupling parameter. The weighting matrix is

$$\mathbf{Q} = \begin{pmatrix} Q_{11} & \epsilon Q_{12} & \cdots & \epsilon Q_{1N} \\ \epsilon Q_{12}^T & Q_{22} & \cdots & \epsilon Q_{2N} \\ \vdots & & \ddots & \vdots \\ \epsilon Q_{1N}^T & \cdots & \epsilon Q_{(N-1)N}^T & Q_{NN} \end{pmatrix}$$

and

$$\mathbf{R} = \text{block diag}\,[R_{11} \quad R_{22} \quad \cdots \quad R_{NN}]$$

The optimal control that minimizes the cost function J is given by (Kwakernaak and Sivan 1972)

$$u_{opt} = -F_{opt}x = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{P}x \tag{3}$$

where $\mathbf{P}$ is the positive semidefinite stabilizing solution of the algebraic Riccati equation

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} + \mathbf{Q} - \mathbf{PSP} = 0, \quad \mathbf{S} = \mathbf{BR}^{-1}\mathbf{B}^T \tag{4}$$

3. Main results

In order to obtain the optimal control (3) in terms of 'ϵ-coupled' subsystems, we present the following recursive fixed point type algorithm to find the solution of the algebraic Riccati equation (4).

Owing to the block dominant structure of matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{Q}$, the solution of

(4) has the following form.

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_{11} & \epsilon\mathbf{P}_{12} & \cdots & \epsilon\mathbf{P}_{1N} \\ \epsilon\mathbf{P}_{12}^{\mathrm{T}} & \mathbf{P}_{22} & \cdots & \epsilon\mathbf{P}_{2N} \\ \vdots & & \ddots & \vdots \\ \epsilon\mathbf{P}_{1N}^{\mathrm{T}} & \cdots & \epsilon\mathbf{P}_{(N-1)N}^{\mathrm{T}} & \mathbf{P}_{NN} \end{pmatrix}$$

Partitioning (4) produces

$$\mathbf{P}_{ii}\mathbf{A}_{ii} + \mathbf{A}_{ii}^{\mathrm{T}}\mathbf{P}_{ii} + \mathbf{Q}_{ii} - \mathbf{P}_{ii}\mathbf{S}_{ii}^{0}\mathbf{P}_{ii} = \epsilon^2 f_{ii} \tag{5}$$

$$\mathbf{D}_i^{\mathrm{T}}\mathbf{P}_{ij} + \mathbf{P}_{ij}\mathbf{D}_j = -\mathbf{Q}_{ij}^{0} + \epsilon f_{ij} \tag{6}$$

where

$$\mathbf{S}_{ii}^{0} = \mathbf{B}_{ii}\mathbf{R}_{ii}^{-1}\mathbf{B}_{ii}^{\mathrm{T}}$$

$$\mathbf{D}_i = \mathbf{A}_{ii} - \mathbf{S}_{ii}^{0}\mathbf{P}_{ii} \tag{7}$$

$$\mathbf{D}_j = \mathbf{A}_{jj} - \mathbf{S}_{jj}^{0}\mathbf{P}_{jj} \tag{8}$$

$$\mathbf{Q}_{ij}^{0} = \mathbf{Q}_{ij} + \mathbf{P}_{ii}\mathbf{A}_{ij} + \mathbf{A}_{ji}^{\mathrm{T}}\mathbf{P}_{jj} - \mathbf{P}_{ii}\mathbf{S}_{ij}^{1}\mathbf{P}_{jj}$$

and

$$f_{ii} = \sum_{l \neq i}^{N} (\mathbf{W}_{il}^{1}\mathbf{P}_{li} - \mathbf{P}_{il}\mathbf{A}_{li} - \mathbf{A}_{li}^{\mathrm{T}}\mathbf{P}_{li}) + \mathbf{W}_{ii}^{2}\mathbf{P}_{ii} + \epsilon\left(\mathbf{W}_{ii}^{3}\mathbf{P}_{ii} + \sum_{l \neq i}^{N} \mathbf{W}_{il}^{2}\mathbf{P}_{li}\right) + \epsilon^2 \sum_{l \neq i}^{N} \mathbf{W}_{il}^{3}\mathbf{P}_{li}$$

$$f_{ij} = \sum_{l \neq i,j}^{N} (\mathbf{W}_{il}^{1}\mathbf{P}_{lj} - \mathbf{P}_{il}\mathbf{A}_{lj} - \mathbf{A}_{li}^{\mathrm{T}}\mathbf{P}_{lj}) + \mathbf{W}_{ij}^{2}\mathbf{P}_{jj} + \epsilon\left(\mathbf{W}_{ii}^{2}\mathbf{P}_{ij} + \mathbf{W}_{ij}^{3}\mathbf{P}_{jj} + \sum_{l \neq i,j}^{N} \mathbf{W}_{il}^{2}\mathbf{P}_{lj}\right) + \epsilon^2 \sum_{l \neq i,j}^{N} \mathbf{W}_{il}^{3}\mathbf{P}_{lj}$$

with

$$\mathbf{W}_{ii}^{2} = \mathbf{P}_{ii}\mathbf{S}_{ii}^{2} + \sum_{l \neq i}^{N} \mathbf{P}_{il}\mathbf{S}_{li}^{1}$$

$$\mathbf{W}_{ii}^{3} = \sum_{l \neq i}^{N} \mathbf{P}_{il}\mathbf{S}_{li}^{2}$$

$$\mathbf{W}_{ij}^{1} = \mathbf{P}_{ii}\mathbf{S}_{ij}^{1} + \mathbf{P}_{ij}\mathbf{S}_{jj}^{0}$$

$$\mathbf{W}_{ij}^{2} = \mathbf{P}_{ii}\mathbf{S}_{ij}^{2} + \sum_{l \neq i,j}^{N} \mathbf{P}_{il}\mathbf{S}_{lj}^{1}$$

$$\mathbf{W}_{ij}^{3} = \mathbf{P}_{ij}\mathbf{S}_{jj}^{2} + \sum_{l \neq i,j}^{N} \mathbf{P}_{il}\mathbf{S}_{lj}^{2}$$

$$\mathbf{S}_{ij}^{1} = \mathbf{B}_{ii}\mathbf{R}_{ii}^{-1}\mathbf{B}_{ji}^{\mathrm{T}} + \mathbf{B}_{ij}\mathbf{R}_{jj}^{-1}\mathbf{B}_{jj}^{\mathrm{T}}$$

$$\mathbf{S}_{ij}^{2} = \sum_{l \neq i,j}^{N} \mathbf{B}_{il}\mathbf{R}_{ll}^{-1}\mathbf{B}_{jl}^{\mathrm{T}}$$

$$\mathbf{S}_{ii}^{2} = \sum_{l \neq i}^{N} \mathbf{B}_{il}\mathbf{R}_{ll}^{-1}\mathbf{B}_{il}^{\mathrm{T}}$$

Since ϵ is a small coupling parameter, we can define $O(\epsilon^2)$ as the approximation of (5) and $O(\epsilon)$ as the approximation of (6) by letting $\epsilon = 0$.

$$\bar{\mathbf{P}}_{ii}\mathbf{A}_{ii} + \mathbf{A}_{ii}^{\mathrm{T}}\bar{\mathbf{P}}_{ii} + \mathbf{Q}_{ii} - \bar{\mathbf{P}}_{ii}\mathbf{S}_{ii}^{0}\bar{\mathbf{P}}_{ii} = 0 \tag{9}$$

$$\bar{\mathbf{D}}_i^{\mathrm{T}}\bar{\mathbf{P}}_{ij} + \bar{\mathbf{P}}_{ij}\bar{\mathbf{D}}_j = -\bar{\mathbf{Q}}_{ij}^{0} \tag{10}$$

where $\bar{\mathbf{D}}_i$, $\bar{\mathbf{D}}_j$ and $\bar{\mathbf{Q}}_{ij}^{0}$ are the matrices defined in (7) and (8) where $\bar{\mathbf{P}}_{ii}$ and $\bar{\mathbf{P}}_{jj}$ are substituted.

The unique positive semidefinite stabilizing solution of (9) (corresponding to subsystem problems) exists under the assumption that the triples $(\mathbf{A}_{ii}, \mathbf{B}_{ii}, \sqrt{\mathbf{Q}_{ii}})$ are stabilizable-detectable. With this assumption, the matrices $\bar{\mathbf{D}}_i$ and $\bar{\mathbf{D}}_j$ are stable (Kwakernaak and Sivan 1972) so that the unique solution of (10) (corresponding to the co-ordinator's problems) also exists.

If the errors caused by approximation are defined as

$$\mathbf{P}_{ii} = \bar{\mathbf{P}}_{ii} + \epsilon^2\mathbf{E}_{ii} \tag{11}$$

$$\mathbf{P}_{ij} = \bar{\mathbf{P}}_{ij} + \epsilon\mathbf{E}_{ij} \quad i, j = 1, \ldots, N, \; i \neq j \tag{12}$$

then the exact solution will be of the form

$$\mathbf{P} = \begin{pmatrix} \bar{\mathbf{P}}_{11} + \epsilon^2\mathbf{E}_{11} & \epsilon(\bar{\mathbf{P}}_{12} + \epsilon\mathbf{E}_{12}) & \cdots & \epsilon(\bar{\mathbf{P}}_{1N} + \epsilon\mathbf{E}_{1N}) \\ \epsilon(\bar{\mathbf{P}}_{12}^{\mathrm{T}} + \epsilon\mathbf{E}_{12}^{\mathrm{T}}) & \bar{\mathbf{P}}_{22} + \epsilon^2\mathbf{E}_{22} & \cdots & \epsilon(\bar{\mathbf{P}}_{2N} + \epsilon\mathbf{E}_{2N}) \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon(\bar{\mathbf{P}}_{1N}^{\mathrm{T}} + \epsilon\mathbf{E}_{1N}^{\mathrm{T}}) & \epsilon(\bar{\mathbf{P}}_{2N}^{\mathrm{T}} + \epsilon\mathbf{E}_{2N}^{\mathrm{T}}) & \cdots & \bar{\mathbf{P}}_{NN} + \epsilon^2\mathbf{E}_{NN} \end{pmatrix} \tag{13}$$

subtracting (9) and (10) from (5) and (6), and using (11) and (12), produces the following equations for the errors.

$$\bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ii} + \mathbf{E}_{ii}\bar{\mathbf{D}}_i = \epsilon^2\mathbf{E}_{ii}\mathbf{S}_{ii}^{0}\mathbf{E}_{ii} + f_{ii} \tag{14}$$

$$\begin{aligned} \bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ij} + \mathbf{E}_{ij}\bar{\mathbf{D}}_j = {} & \epsilon\mathbf{E}_{ii}\mathbf{S}_{ii}^{0}\mathbf{P}_{ij} + \epsilon\mathbf{P}_{ij}\mathbf{S}_{jj}^{0}\mathbf{E}_{jj} - \epsilon\mathbf{E}_{ii}\mathbf{A}_{ij} - \epsilon\mathbf{A}_{ji}^{\mathrm{T}}\mathbf{E}_{jj} \\ & + \epsilon\mathbf{E}_{ii}\mathbf{S}_{ij}^{1}\mathbf{P}_{jj} + \epsilon\mathbf{P}_{ii}\mathbf{S}_{ij}^{1}\mathbf{E}_{jj} + f_{ij} - \epsilon^3\mathbf{E}_{ii}\mathbf{S}_{ij}^{1}\mathbf{E}_{jj} \end{aligned} \tag{15}$$

It can be shown easily that the nonlinear eqns. (14) and (15) have the form

$$\bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ii} + \mathbf{E}_{ii}\bar{\mathbf{D}}_i = \text{const} + \epsilon^2F(\mathbf{E}_{ii}) \tag{16}$$

$$\bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ij} + \mathbf{E}_{ij}\bar{\mathbf{D}}_j = \text{const} + \epsilon F(\mathbf{E}_{ii}, \mathbf{E}_{jj}, \mathbf{E}_{ij}) \tag{17}$$

We can see that all nonlinear terms of error are multiplied by ϵ^2 in (14), and ϵ in (15), so that we propose the following reduced-order fixed point parallel algorithm to solve eqns. (14) and (15).

$$\bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ii}^{(m+1)} + \mathbf{E}_{ii}^{(m+1)}\bar{\mathbf{D}}_i = \epsilon^2\mathbf{E}_{ii}^{(m)}\mathbf{S}_{ii}^{0}\mathbf{E}_{ii}^{(m)} + f_{ii}^{(m)} \tag{18}$$

$$\begin{aligned} \bar{\mathbf{D}}_i^{\mathrm{T}}\mathbf{E}_{ij}^{(m+1)} + \mathbf{E}_{ij}^{(m+1)}\bar{\mathbf{D}}_j = {} & \epsilon[\mathbf{E}_{ii}^{(m+1)}\mathbf{S}_{ii}^{0}\mathbf{P}_{ij}^{(m)} + \mathbf{P}_{ij}^{(m)}\mathbf{S}_{jj}^{0}\mathbf{E}_{jj}^{(m+1)} - \mathbf{E}_{ii}^{(m+1)}\mathbf{A}_{ij} - \mathbf{A}_{ji}^{\mathrm{T}}\mathbf{E}_{jj}^{(m+1)} \\ & + \mathbf{E}_{ii}^{(m+1)}\mathbf{S}_{ij}^{1}\mathbf{P}_{jj}^{(m+1)} + \mathbf{P}_{ii}^{(m+1)}\mathbf{S}_{ij}^{1}\mathbf{E}_{jj}^{(m+1)}] + f_{ij}(\mathbf{P}_{ii}^{(m+1)}, \mathbf{P}_{ij}^{(m)}) \end{aligned} \tag{19}$$

with $\mathbf{E}_{ii}^{(0)} = 0$, $\mathbf{E}_{ij}^{(0)} = 0$, where

$$\mathbf{P}_{ii}^{(m)} = \bar{\mathbf{P}}_{ii} + \epsilon^2 \mathbf{E}_{ii}^{(m)} \tag{20}$$

$$\mathbf{P}_{ij}^{(m)} = \bar{\mathbf{P}}_{ij} + \epsilon \mathbf{E}_{ij}^{(m)} \tag{21}$$

$$i, j = 1, 2, \ldots, N, \; i \neq j, \quad m = 0, 1, 2, \ldots$$

Theorem 1

Under the assumption stated above, the solutions of algorithms (18) and (19) converge to the exact solution of $\mathbf{E}_{ii}$ with the rate of $O(\epsilon^2)$, and $\mathbf{E}_{ij}$ with the rate of $O(\epsilon)$; that is

$$\|\mathbf{E}_{ii} - \mathbf{E}_{ii}^{(m)}\| = O(\epsilon^{2m}) \tag{22}$$

$$\|\mathbf{E}_{ij} - \mathbf{E}_{ij}^{(m)}\| = O(\epsilon^{m}) \tag{23}$$

Proof

The Jacobian of (14) and (15) at some $\epsilon = 0$ is given

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_{ii} & 0 \\ * & \mathbf{J}_{ij} \end{pmatrix} \tag{24}$$

where

$$\mathbf{J}_{ii} = \mathbf{I}_i \otimes \mathbf{D}_i^{\mathrm{T}} + \mathbf{D}_i^{\mathrm{T}} \otimes \mathbf{I}_i \tag{25}$$

$$\mathbf{J}_{ij} = \mathbf{I}_j \otimes \mathbf{D}_j^{\mathrm{T}} + \mathbf{D}_i^{\mathrm{T}} \otimes \mathbf{I}_i \tag{26}$$

Since $\mathbf{D}_i$ and $\mathbf{D}_j$ are stable matrices (by the assumption), $\mathbf{J}_{ii}$ and $\mathbf{J}_{ij}$ are non-singular and hence the Jacobian will be non-singular at $\epsilon = 0$. By the implicity function theorem, the existence of the unique bounded solution of (14) and (15) is guaranteed for sufficiently small values of ϵ.

Next we try to find an estimate of the rate of convergence. From (18) and (19) it is easy to show that

$$(\mathbf{E}_{ii}^{(m+1)} - \mathbf{E}_{ii}^{(m)})\bar{\mathbf{D}}_i + \bar{\mathbf{D}}_i^{\mathrm{T}}(\mathbf{E}_{ii}^{(m+1)} - \mathbf{E}_{ii}^{(m)}) = \epsilon^2 h_{ii}(\mathbf{E}^{(m)}) \tag{27}$$

$$(\mathbf{E}_{ij}^{(m+1)} - \mathbf{E}_{ij}^{(m)})\bar{\mathbf{D}}_j + \bar{\mathbf{D}}_i^{\mathrm{T}}(\mathbf{E}_{ij}^{(m+1)} - \mathbf{E}_{ij}^{(m)}) = \epsilon h_{ij}(\mathbf{E}^{(m)}) \tag{28}$$

where $h_{ii}(\mathbf{E}^{(m)})$ and $h_{ij}(\mathbf{E}^{(m)})$ are functions of $\mathbf{E}_{ii}^{(m)}$ and $\mathbf{E}_{ij}^{(m)}$.

From (27) and (28) we can conclude that

$$\|\mathbf{E}_{ii} - \mathbf{E}_{ii}^{(m)}\| = O(\epsilon^{2m}) \tag{29}$$

$$\|\mathbf{E}_{ij} - \mathbf{E}_{ij}^{(m)}\| = O(\epsilon^{m}) \tag{30}$$

To obtain a solution of the Riccati equation, $\mathbf{P}$ (which has dimensions $n \times n = (n_1 + n_2 + \cdots + n_N) \times (n_1 + n_2 + \cdots + n_N)$), we only need to solve N reduced-order algebraic Riccati equations of dimensions $(n_i \times n_i)$, and $N(N-1)/2 - N$ reduced-order algebraic Lyapunov equations of dimensions $(n_i \times n_j)$, $i, j = 1, 2, \ldots, N$; $i \neq j$, separately or in parallel. After obtaining the solution $\mathbf{P}^{(m)}$ of the Riccati equation, the optimal control is then

$$u_{\mathrm{opt}}^{(m)} = -\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}^{(m)}x \tag{31}$$

4. Numerical example

In order to illustrate the efficiency of the proposed algorithm for weakly coupled systems, the algorithm is applied to a power system with three plants as depicted in the Figure (Delacour *et al.* 1978).

The system matrices **A** and **B** are

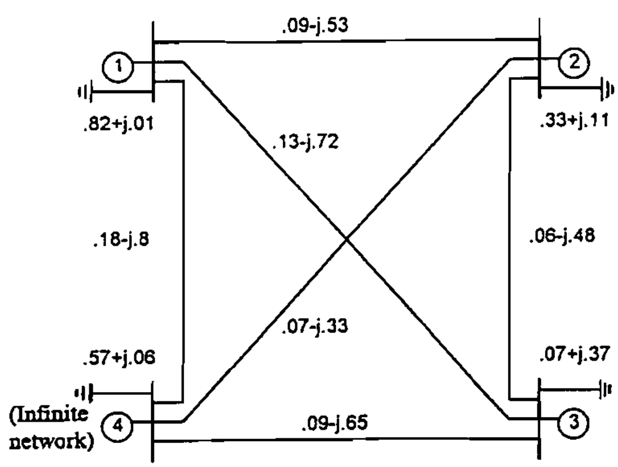
$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \epsilon\mathbf{A}_{12} & \epsilon\mathbf{A}_{13} \\ \epsilon\mathbf{A}_{21} & \mathbf{A}_{22} & \epsilon\mathbf{A}_{23} \\ \epsilon\mathbf{A}_{31} & \epsilon\mathbf{A}_{32} & \mathbf{A}_{33} \end{pmatrix} \tag{32}$$

$$\mathbf{A}_{11} = \begin{pmatrix} 0{\cdot}0 & 1{\cdot}0 & -0{\cdot}266 & -0{\cdot}009 \\ -2{\cdot}75 & -2{\cdot}78 & -1{\cdot}36 & -0{\cdot}037 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 1{\cdot}0 \\ -4{\cdot}95 & 0{\cdot}0 & -55{\cdot}5 & -0{\cdot}039 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{12} = \begin{pmatrix} 0{\cdot}0024 & 0{\cdot}0 & -0{\cdot}087 & 0{\cdot}002 \\ -0{\cdot}185 & 0{\cdot}0 & 1{\cdot}11 & -0{\cdot}011 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}222 & 0{\cdot}0 & 8{\cdot}17 & 0{\cdot}004 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{13} = \begin{pmatrix} 0{\cdot}073 & 0{\cdot}0 & -0{\cdot}25 & 0{\cdot}003 \\ -0{\cdot}46 & 0{\cdot}0 & 2{\cdot}8 & -0{\cdot}02 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ 0{\cdot}924 & 0{\cdot}0 & 17{\cdot}5 & 0{\cdot}02 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{21} = \begin{pmatrix} 0{\cdot}021 & 0{\cdot}0 & 0{\cdot}121 & 0{\cdot}003 \\ -1{\cdot}1 & 0{\cdot}0 & -1{\cdot}62 & -0{\cdot}015 \\ 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 & 0{\cdot}0 \\ -2{\cdot}43 & 0{\cdot}0 & 1{\cdot}37 & -0{\cdot}034 \end{pmatrix}$$



Typical three-machine power system.

$$\mathbf{A}_{22} = \begin{pmatrix} -0.21 & 1.0 & -1.6 & -0.005 \\ -1.9 & -1.8 & 9.3 & -0.12 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ -3.1 & 0.0 & -56 & 0.032 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{23} = \begin{pmatrix} 0.06 & 0.0 & 0.46 & 0.002 \\ -1.0 & 0.0 & 1.49 & -0.04 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.12 & 0.0 & 29.8 & -0.028 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{31} = \begin{pmatrix} -0.002 & 0.0 & 0.83 & 0.0 \\ -6.78 & 0.0 & -10.1 & 0.09 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ -1.24 & 0.0 & 0.498 & -0.017 \end{pmatrix}$$

$$\epsilon\mathbf{A}_{32} = \begin{pmatrix} 0.011 & 0.0 & 0.22 & 0.0 \\ -2.1 & 0.0 & 1.7 & -0.123 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ -0.07 & 0.0 & 6.38 & -0.011 \end{pmatrix}$$

$$\mathbf{A}_{33} = \begin{pmatrix} -0.197 & 1.0 & -1.2 & -0.003 \\ -54.5 & -20.0 & 70.1 & -2.37 \\ 0.0 & 0.0 & 0.0 & 1.0 \\ -3.4 & 0.0 & -21.0 & -0.017 \end{pmatrix}$$

and

$$\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 0 & 36.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 78.9 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1000 & 0 & 0 \end{pmatrix}$$

The weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are chosen

$$\mathbf{Q} = \mathbf{I}_{12\times 12}, \quad \mathbf{R} = \mathbf{I}_{3\times 3}$$

The small coupling parameters are built into the problem of the value of $\epsilon = 29.948/59.132 = 0.5065$, which is estimated from the strongest coupled row (8th row from matrix $\mathbf{A}$). Simulation results are obtained with the provided algorithm by using the package Matlab (Hill 1988).

The zeroth-order solution of the Riccati equation

$$\mathbf{P}_{11}^{(0)} = \begin{pmatrix} 1.7110 & 0.0433 & 7.1686 & -0.1702 \\ 0.0433 & 0.0268 & 0.1881 & 0.0004 \\ 7.1686 & 0.1881 & 80.0861 & -0.4452 \\ -0.1702 & 0.0004 & -0.4452 & 1.4409 \end{pmatrix}$$

$$\mathbf{P}_{22}^{(0)} = \begin{pmatrix} 1\cdot2688 & 0\cdot0155 & 6\cdot6411 & -0\cdot1766 \\ 0\cdot0155 & 0\cdot0126 & 0\cdot0848 & -0\cdot0012 \\ 6\cdot6411 & 0\cdot0848 & 121\cdot6397 & -0\cdot5668 \\ -0\cdot1766 & -0\cdot0012 & -0\cdot5668 & 2\cdot1869 \end{pmatrix}$$

$$\mathbf{P}_{33}^{(0)} = \begin{pmatrix} 1\cdot5579 & 0\cdot0015 & 3\cdot3665 & -0\cdot2993 \\ 0\cdot0015 & 0\cdot0010 & 0\cdot0034 & -0\cdot0003 \\ 3\cdot3665 & 0\cdot0034 & 22\cdot6512 & -0\cdot4379 \\ -0\cdot2993 & -0\cdot0003 & -0\cdot4379 & 1\cdot1429 \end{pmatrix}$$

$$\mathbf{P}_{12}^{(0)} = \begin{pmatrix} 0\cdot3739 & 0\cdot0036 & 3\cdot7007 & -1\cdot6437 \\ 0\cdot0126 & 0\cdot0001 & 0\cdot1568 & -0\cdot0394 \\ -1\cdot9740 & -0\cdot0334 & -63\cdot1022 & -16\cdot7590 \\ 0\cdot9571 & 0\cdot0115 & 16\cdot6150 & -0\cdot9069 \end{pmatrix}$$

$$\mathbf{P}_{13}^{(0)} = \begin{pmatrix} 0\cdot5348 & 0\cdot0005 & -0\cdot6635 & -0\cdot5101 \\ 0\cdot0142 & 0\cdot0000 & -0\cdot0067 & -0\cdot0135 \\ -5\cdot7676 & -0\cdot0058 & -67\cdot9798 & -1\cdot5919 \\ 0\cdot2803 & 0\cdot0003 & 1\cdot3201 & -1\cdot1969 \end{pmatrix}$$

$$\mathbf{P}_{23}^{(0)} = \begin{pmatrix} -0\cdot0347 & 0\cdot0000 & -8\cdot5453 & -0\cdot3252 \\ -0\cdot0005 & 0\cdot0000 & -0\cdot1052 & -0\cdot0054 \\ -8\cdot6134 & -0\cdot0086 & -181\cdot4635 & -3\cdot7095 \\ 0\cdot4694 & 0\cdot0005 & 3\cdot6767 & -2\cdot6878 \end{pmatrix}$$

Near-optimum solution of the Riccati equation

After six iterations we obtain the solution of the Riccati equation $\mathbf{P}^{(6)}$ which is equal to the exact solution $\mathbf{P}$ with an accuracy to five places of decimals

$$\mathbf{P}_{11}^{(6)} = \begin{pmatrix} 1\cdot7846 & 0\cdot0454 & 5\cdot4494 & 0\cdot0133 \\ 0\cdot0454 & 0\cdot0269 & 0\cdot1372 & 0\cdot0039 \\ 5\cdot4494 & 0\cdot1372 & 74\cdot1546 & -0\cdot4305 \\ 0\cdot0133 & 0\cdot0039 & -0\cdot4305 & 1\cdot3143 \end{pmatrix}$$

$$\mathbf{P}_{22}^{(6)} = \begin{pmatrix} 1\cdot2566 & 0\cdot0154 & 6\cdot6222 & -0\cdot1569 \\ 0\cdot0154 & 0\cdot0126 & 0\cdot0844 & -0\cdot0009 \\ 6\cdot6222 & 0\cdot0844 & 131\cdot1519 & -0\cdot4683 \\ -0\cdot1569 & -0\cdot0009 & -0\cdot4683 & 2\cdot2080 \end{pmatrix}$$

$$\mathbf{P}_{33}^{(6)} = \begin{pmatrix} 2\cdot1011 & 0\cdot0020 & 8\cdot1844 & -0\cdot6244 \\ 0\cdot0020 & 0\cdot0010 & 0\cdot0083 & -0\cdot0006 \\ 8\cdot1844 & 0\cdot0083 & 97\cdot9280 & -0\cdot2530 \\ -0\cdot6244 & -0\cdot0006 & -0\cdot2530 & 2\cdot6308 \end{pmatrix}$$

$$\mathbf{P}_{12}^{(6)} = \begin{pmatrix} 0.4614 & 0.0051 & 3.7935 & -0.7849 \\ 0.0131 & 0.0002 & 0.1260 & -0.0175 \\ -1.6268 & -0.0252 & -40.3198 & -8.8318 \\ 0.4813 & 0.0057 & 8.7481 & -0.5496 \end{pmatrix}$$

$$\mathbf{P}_{13}^{(6)} = \begin{pmatrix} 0.2585 & 0.0002 & -5.5951 & -0.1508 \\ 0.0068 & 0.0000 & -0.1508 & -0.0078 \\ -6.1631 & -0.0062 & -54.9269 & 5.2550 \\ -0.1026 & -0.0001 & -5.4605 & -1.0543 \end{pmatrix}$$

$$\mathbf{P}_{23}^{(6)} = \begin{pmatrix} -0.0643 & -0.0001 & -7.8261 & -0.3901 \\ -0.0008 & 0.0000 & -0.0946 & -0.0061 \\ -6.9247 & -0.0070 & -163.9658 & -6.5772 \\ 0.7012 & 0.0007 & 6.5096 & -2.4871 \end{pmatrix}$$

Using (31) we can obtain the optimal control

$$\begin{aligned} \boldsymbol{u}_1 = {} & [-1.6387 \quad -0.9702 \quad -4.9532 \quad -0.1411]\boldsymbol{x}_1 \\ & + [-0.2390 \quad -0.0028 \quad -2.3036 \quad 0.3199]\boldsymbol{x}_2 \\ & + [-0.1245 \quad -0.0001 \quad 2.7582 \quad 0.1424]\boldsymbol{x}_3 \end{aligned} \tag{33}$$

$$\begin{aligned} \boldsymbol{u}_2 = {} & [-0.2040 \quad -0.0062 \quad 1.0053 \quad -0.2269]\boldsymbol{x}_1 \\ & + [-1.2136 \quad -0.9927 \quad -6.6582 \quad 0.0720]\boldsymbol{x}_2 \\ & + [0.0301 \quad 0.000 \quad 3.7810 \quad 0.2419]\boldsymbol{x}_3 \end{aligned} \tag{34}$$

$$\begin{aligned} \boldsymbol{u}_3 = {} & [-0.1239 \quad -0.0034 \quad 3.1218 \quad 0.0549]\boldsymbol{x}_1 \\ & + [0.0356 \quad 0.0004 \quad 3.5214 \quad -0.3508]\boldsymbol{x}_2 \\ & + [-2.0447 \quad -0.9822 \quad -8.2549 \quad 0.6171]\boldsymbol{x}_3 \end{aligned} \tag{35}$$

5. Conclusion

A recursive reduced-order parallel algorithm has been developed to find the optimal control of large-scale systems. The algorithm is based on the fixed point approach to a small coupling parameter problem, where the small coupling parameter plays the role of the radius of the converge. It has been shown that the algorithm is computationally very efficient, especially when a high order of accuracy is required.

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