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Robust estimation and compensation for actuator and sensor failures in linear systems

Q. XIA†, M. RAO†, S. X. SHEN‡ and V.-G. GOURISHANKAR‡

A computationally feasible technique for the robust detection and estimation for actuator and sensor failures is presented. Model errors and component failures are represented by a bias vector called the failure state in system and measurement equations. A Kalman–Bucy filter is implemented to estimate the system state, and to generate the corresponding residuals. These residuals are then processed by using an adaptive fading Kalman filter to give the failure state estimate. The final state estimate is obtained by compensating model errors and component failures in the filter based on no failure assumption. The divergency of the filter based on no failure assumption is avoided by stepwise compensation of the failure state. The technique is applicable to the detection, estimation and compensation of slowly varying model errors and suddenly occurring component failures, and to the discrimination between them.

1. Introduction

The Kalman–Bucy filter works extremely well in cases where the dynamical system is linear and accurately modelled. However, most real systems include model errors, arising from variation of parameters, nonlinearities and/or noise. Moreover, some system components, such as actuators and sensors, may fail. To improve the reliability and performance of the control system, a filter which is able to determine the extent of failures and the time of occurrence, and to compensate for them, is required.

Sensor and actuator failures will result in abrupt changes in system dynamics. In recent years many methods and structures have been proposed to detect these changes (Willsky 1976, Deckert *et al.* 1977, Iserman 1984). The two basic and extremely important structures are the multiple filter structure and residual-based structure. The failure detection techniques of multiple filter structure are robust to model parameters. However, the numerical burden may be too large, and estimation of the extent of the failure is quite difficult. The failure detection techniques of residual-based structures usually need less computation, but are sensitive to model parameters (Willsky and Jones 1976).

Two failure detection techques have been developed using the separated-bias estimation algorithm introduced by Friedland (1969, 1978). The separated-bias estimation algorithm is exact when the bias is constant. It does not apply rigorously, however, to detect sudden changes, since it takes a long period of time to estimate the extent of failure accurately. Friedland and Grabousky (1980) provided an effective technique for the detection of failures by combining the separated-bias estimation algorithm with the maximum-likelihood failure detection and estimation method.

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The technique is, however, too complicated. Caglayan (1980) gives a computationally feasible algorithm by applying the separated-bias estimation results to the parameter adaptive estimation solution, but the failure state estimate cannot track its true value rapidly, and the filter based on the assumption of no failure may diverge (Ignagni 1981).

In this paper, a new method is developed for failure detection, estimation and compensation. The separated-bias estimation algorithm is combined with the adaptive fading Kalman filter developed by Xia *et al.* (1992) to give a simple and effective detection technique for actuator and sensor failures. The failure states (biases) represent component failures when actuators and sensors have failed, or model errors when there is no failure. The correction of the filter based on the no-failure assumption is performed stepwisely to avoid the filter to diverge.

2. Failure model of system

Consider a linear, discrete-time, stochasic system

$$\boldsymbol{x}(k+1) = \mathbf{A}(k)\boldsymbol{x}(k) + \mathbf{B}(k)\boldsymbol{u}(k) + \boldsymbol{w}(k) \tag{1}$$

$$\boldsymbol{y}(k) = \mathbf{H}(k)\boldsymbol{x}(k) + \boldsymbol{v}(k) \tag{2}$$

where $\boldsymbol{x}(k) \in \mathbb{R}^n$, $\boldsymbol{u}(k) \in \mathbb{R}^q$ and $\boldsymbol{y}(k) \in \mathbb{R}^m$ are the state, control and measurement, respectively, $\boldsymbol{w}(k)$ and $\boldsymbol{v}(k)$ denote sequences of uncorrelated gaussian random vectors with zero means and covariance matrices $\mathbf{Q}(k)$ and $\mathbf{R}(k)$, respectively. The initial state $\boldsymbol{x}(0)$ is specified as a random gaussian vector with mean $\tilde{\boldsymbol{x}}(0)$ and covariance $\tilde{\mathbf{P}}(0)$.

We assume that system (1) and (2) is completely observable. The equations which describe the optimal state estimator are given by (Maybeck 1982)

$$\tilde{\boldsymbol{x}}(k|k-1) = \mathbf{A}(k-1)\tilde{\boldsymbol{x}}(k-1) + \mathbf{B}(k-1)\boldsymbol{u}(k-1) \tag{3}$$

$$\tilde{\boldsymbol{x}}(k) = \tilde{\boldsymbol{x}}(k|k-1) + \mathbf{K}(k)\tilde{\boldsymbol{\gamma}}(k) \tag{4}$$

where $\tilde{\boldsymbol{\gamma}}(k)$ is the measurement residual

$$\tilde{\boldsymbol{\gamma}}(k) = \boldsymbol{y}(k) - \mathbf{H}(k)\tilde{\boldsymbol{x}}(k|k-1) \tag{5}$$

The gain and error covariance satisfy

$$\left.\begin{aligned} \mathbf{K}(k) &= \tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k)[\mathbf{H}(k)\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k)]^{-1} \\ \tilde{\mathbf{P}}(k|k-1) &= \mathbf{A}(k-1)\tilde{\mathbf{P}}(k)\mathbf{A}^{\mathrm{T}}(k-1) + \mathbf{Q}(k-1) \\ \tilde{\mathbf{P}}(k) &= [\mathbf{I} - \mathbf{K}(k)\mathbf{H}(k)]\tilde{\mathbf{P}}(k|k-1) \end{aligned}\right\} \tag{6}$$

This filter is quite useful to monitor the system performance, and $\tilde{\boldsymbol{\gamma}}(k)$ is the zero-mean white noise sequence with the covariance $\mathbf{C}_0(k)$ if the system model is perfect

$$\mathbf{C}_0(k) = \mathbf{H}(k)\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k) \tag{7}$$

However, if there are model errors or component failures, the performances of the filter will be greatly degraded. Now we discuss the effect of actuator and sensor failures on the performance of the filter.

Denoting the control actions of the ith actuator in normal and fault conditions by $\boldsymbol{u}_i(k)$ and $\tilde{\boldsymbol{u}}_i(k)$, respectively, and by defining

$$\boldsymbol{d}_{ai}(k) = \tilde{\boldsymbol{u}}_i(k) - \boldsymbol{u}_i(k) \tag{8}$$

then the dynamics of the system (1) and (2) with failure in the ith actuator is

$$\boldsymbol{x}(k+1) = \mathbf{A}(k)\boldsymbol{x}(k) + \mathbf{B}(k)\boldsymbol{u}(k) + \boldsymbol{b}_i(k)\boldsymbol{d}_{ai}(k) + \boldsymbol{w}(k) \tag{9}$$

Here $\boldsymbol{b}_i(k)$ is the ith column of the matrix $\mathbf{B}(k)$.

Similarly, by denoting the outputs of the jth sensor in normal and fault conditions by $y_j(k)$ and $\tilde{y}_j(k)$, respectively, and by defining

$$\boldsymbol{d}_{sj}(k) = \tilde{y}_j(k) - y_j(k) \tag{10}$$

the measurement equation with failure in the jth sensor is then

$$\boldsymbol{y}(k) = \mathbf{H}(k)\boldsymbol{x}(k) + \boldsymbol{e}_j(k)\boldsymbol{d}_{sj}(k) + \boldsymbol{v}(k) \tag{11}$$

Here $\boldsymbol{e}_j(k)$ is an m-dimensional vector with all elements zero, except the jth element which is one.

Assume, with no loss of generality, that the 1st, 2nd, ..., $\bar{q}$th actuators and the 1st, 2nd, ..., $\bar{m}$th sensors may fail in operation, then a p-dimensional vector, called the failure state, can be defined as

$$\boldsymbol{d}(k) = [\boldsymbol{d}_{a1}(k) \quad \dots \quad \boldsymbol{d}_{a\bar{q}}(k) \quad \boldsymbol{d}_{s1}(k) \quad \dots \quad \boldsymbol{d}_{s\bar{m}}(k)]^{\mathrm{T}} \tag{12}$$

where $\bar{q} \le q$, $\bar{m} \le m$, $p = \bar{q} + \bar{m}$. The system dynamics and measurement equations become

$$\boldsymbol{x}(k+1) = \mathbf{A}(k)\boldsymbol{x}(k) + \mathbf{B}(k)\boldsymbol{u}(k) + \mathbf{E}_1(k)\boldsymbol{d}(k) + \boldsymbol{w}(k) \tag{13}$$

$$\boldsymbol{y}(k) = \mathbf{H}(k)\boldsymbol{x}(k) + \mathbf{E}_2(k)\boldsymbol{d}(k) + \boldsymbol{v}(k) \tag{14}$$

where $\mathbf{E}_1(k) \in \mathbb{R}^{n\times p}$ and $\mathbf{E}_2(k) \in \mathbb{R}^{m\times p}$ are coefficient matrices of the failure state given by

$$\left.\begin{aligned} \mathbf{E}_1(k) &= [\boldsymbol{b}_1(k) \quad \dots \quad \boldsymbol{b}_{\bar{q}}(k) \quad 0 \quad \dots \quad 0] \\ \mathbf{E}_2(k) &= [0 \quad \dots \quad 0 \quad \boldsymbol{e}_1(k) \quad \dots \quad \boldsymbol{e}_{\bar{m}}(k)] \end{aligned}\right\} \tag{15}$$

According to the discussion above it is obvious that actuator and sensor failures will result in a failure state vector (biases) in system and measurement equations. An actuator or sensor may completely fail or simply suffer degradation in performance, in the form of bias or increased inaccuracies. In this paper we assume that the control actions of actuators and the output of sensors contain biases when they have failed. The Kalman filter could not adapt to large unmodelled phenomena, i.e. the large biases introduced by model errors and component failures, and the estimation performance will be degraded. Some techniques which detect and estimate the failures and compensate them in state estimation, are required.

3. Adaptive fading Kalman filter

Before going further into the failure detection and estimation problem, let us first recall the adaptive fading Kalman filter developed by Xia *et al.* (1992).

The equations describing the adaptive fading Kalman filter are identical in form to those of the normal Kalman filter given in (3)–(6), except for the forgetting factor $\lambda(k)$ in the time propagation error covariance equation (6), that is

$$\tilde{\mathbf{P}}(k|k-1) = \lambda(k)\mathbf{A}(k)\tilde{\mathbf{P}}(k-1)^{\mathrm{T}}(k) + \mathbf{Q}(k) \tag{16}$$

where $\lambda(k) \ge 1$.

The performance of the filter depends on the correct choice of forgetting factor. The adaptive fading Kalman filter adjusts the forgetting factor $\lambda(k)$ by using measured outputs to improve the optimality and convergence of the Kalman filter. Rapid fading occurs when data give poor fit to the model, and slow fading for good fit.

It is shown (Xia *et al.* 1992) that three possible choices for $\lambda(k)$ will yield optimally adaptive forgetting phenomena. These choices are obtained based on a property of the Kalman filter: the sequence of residuals is uncorrelated when the optimal gain is used.

The autocovariance matrix of $\tilde{\gamma}(k)$ is given by (Ohap and Stubberud 1976)

$$\begin{aligned}\mathbf{C}_j(k) &= \mathbf{E}[\tilde{\gamma}(k+j)\tilde{\gamma}^{\mathrm{T}}(k)]\\ &= \mathbf{H}(k+j)\mathbf{A}(k+j-1)[\mathbf{I}-\mathbf{K}(k+j-1)\mathbf{H}(k+j-1)] \times \cdots\\ &\quad \times \mathbf{A}(k+1)[\mathbf{I}-\mathbf{K}(k+1)\mathbf{H}(k+1)]\mathbf{A}(k)\\ &\quad \times [\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k)-\mathbf{K}(k)\mathbf{C}_0(k)], \quad j \neq 0\end{aligned} \tag{17}$$

Since the sequence of residuals of optimal state estimator is uncorrelated, $\mathbf{C}_j(k)$ must be identically zero. It is obvious that if a forgetting factor $\lambda(k)$ can be chosen such that

$$\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k)-\mathbf{K}(k)\mathbf{C}_0(k)=0 \tag{18}$$

then $\mathbf{K}(k)$ is optimal. Defining

$$\mathbf{Z}(k)=\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k)-\mathbf{K}(k)\mathbf{C}_0(k) \tag{19}$$

the optimality of the Kalman filter can be judged by a scalar function $f(k)$

$$f(k)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{m}\mathbf{Z}_{ij}^2(k) \tag{20}$$

Hence, the forgetting factor should be chosen to satisfy (18) or to minimize $f(k)$. This forms the basis of choosing $\lambda(k)$.

It should be noted that $\mathbf{C}_0(k)$ in (18) and (19) is computed from measured data, rather than from the theoretical equations (6) and (7).

Proposition 1

Given the system state equations (1) and (2) with the following two conditions satisfied—$\mathbf{Q}(k)$, $\mathbf{R}(k)$ and $\mathbf{P}(0)$ are positive definite matrices for all k; and the measurement matrix $\mathbf{H}(k)$ is fully ranked for all k—then the optimal forgetting factor is computed by

$$\lambda(k)=\max\left\{1, \frac{1}{m}\operatorname{trace}[\mathbf{N}(k)\mathbf{M}^{-1}(k)]\right\} \tag{21}$$

where

$$\left.\begin{aligned}\mathbf{M}(k)&=\mathbf{H}(k)\mathbf{A}(k)\tilde{\mathbf{P}}(k-1)\mathbf{A}^{\mathrm{T}}(k)\mathbf{H}^{\mathrm{T}}(k)\\ \mathbf{N}(k)&=\mathbf{C}_0(k)-\mathbf{H}(k)\mathbf{Q}(k)\mathbf{H}^{\mathrm{T}}(k)-\mathbf{R}(k)\end{aligned}\right\} \tag{22}$$

Proposition 2

System state equations and conditions are the same as in Proposition 1. The optimal forgetting factor can be computed from

$$\lambda(k)=\max\{1, \operatorname{trace}[\mathbf{N}(k)]/\operatorname{trace}[\mathbf{M}(k)]\} \tag{23}$$

where the matrices $\mathbf{M}(k)$ and $\mathbf{N}(k)$ are given in (22).

Proposition 3

Given the system equations (1) and (2), the optimal forgetting factor can be obtained by iterative computation

$$\lambda^{l+1}(k) = \lambda^l(k) - \phi \frac{\partial f^l(k)}{\partial \lambda^l(k)} \tag{24}$$

with the initial conditions

$$\lambda^0(1) = 1, \quad \lambda^0(k) = \lambda(k-1) \tag{25}$$

where the superscript l denotes the number of iterations at a time instant; ϕ is step length. If the following condition holds at the pth iteration

$$|\lambda^p(k) - \lambda^{p-1}(k)| < \epsilon, \quad \epsilon > 0 \tag{26}$$

then stop the iteration and let

$$\lambda(k) = \lambda^p(k) \tag{27}$$

The gradient term in (24) is

$$\frac{\partial f^l(k)}{\partial \lambda^l(k)} = \sum_{i=1}^{n} \sum_{j=1}^{m} Z^l_{ij}(k) \left[\frac{\partial Z^l(k)}{\partial \lambda^l(k)}\right]_{ij} \tag{28}$$

$$\begin{aligned}\frac{\partial Z^l(k)}{\partial \lambda^l(k)} &= \mathbf{A}(k)\tilde{\mathbf{P}}(k-1)\mathbf{A}^{\mathrm{T}}(k)\mathbf{H}^{\mathrm{T}}(k)\{\mathbf{I} - [\mathbf{T}^l(k)]^{-1}\mathbf{C}_0(k)\} \\ &\quad + \mathbf{K}^l(k)\mathbf{H}(k)\mathbf{A}(k)\tilde{\mathbf{P}}(k-1)\mathbf{A}^{\mathrm{T}}(k)\mathbf{H}^{\mathrm{T}}(k)\{\mathbf{I} + [\mathbf{T}^l(k)]^{-1}\mathbf{C}_0(k)\}\end{aligned} \tag{29}$$

$$\mathbf{T}^l(k) = \mathbf{H}(k)\tilde{\mathbf{P}}^l(k|k-1)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k) \tag{30}$$

where $\tilde{\mathbf{P}}^l(k)$, $\mathbf{K}^l(k)$ and $\mathbf{Z}^l(k)$ are the corresponding matrices when $\lambda^l(k)$ is used.

Proposition 4

The covariance of $\gamma(k)$ can be estimated by a recursive fading formula

$$\mathbf{C}_0(k) = \frac{\mathbf{G}_1(k)}{\mathbf{G}_2(k)} \tag{31}$$

where

$$\left.\begin{aligned}\mathbf{G}_1(k) &= \mathbf{G}_1(k-1)/\lambda(k-1) + \tilde{\gamma}(k)\tilde{\gamma}^{\mathrm{T}}(k) \\ \mathbf{G}_2(k) &= \mathbf{G}_2(k-1)/\lambda(k-1) + 1\end{aligned}\right\} \tag{32}$$

with the initial conditions

$$\mathbf{G}_1(0) = 0, \quad \mathbf{G}_2(0) = 0$$

In the next section the adaptive fading Kalman filter will be applied to detect and estimate the failure state $d(k)$.

4. Simultaneous failure detection and estimation

In this paper the failure state $\boldsymbol{d}(k)$ represents both model errors and component failures. In the former case $\boldsymbol{d}(k)$ varies slowly, but in the latter case $\boldsymbol{d}(k)$ is subjected to sudden changes. The algorithm used to detect and estimate $d(k)$ must be applicable to both cases. In Friedland's separated-bias algorithm (Friedland 1969, 1978) the filter-computed error covariance matrix and the gain matrix of bias estimate decrease to reach lower bounds with time. Since the estimation of the filter is not affected significantly by the most recent measured data, Friedland's algorithm is exact only when $\boldsymbol{d}(k)$ is a constant bias. It cannot respond quickly enough to a sudden change, such as a component failure.

The adaptive fading Kalman filter can automatically and quickly adjust the forgetting factor and thus the gain matrix. It can therefore respond quickly to a sudden change. This fact suggests a practical failure detection and estimation strategy: process the measurement equation (2) by a filter based on no failure assumption and generate the corresponding residuals. These residuals are processed to estimate the failure state $\boldsymbol{d}(k)$. When no component has failed the residuals are small and the gain matrix of failure state estimator remains in a relatively low level. Hence the estimate of $\boldsymbol{d}(k)$ is suitable to track slowly varying model errors. When some actuators and sensors have failed, the increasing residuals cause the gain matrix of the failure state estimate to increase quickly by adjusting the forgetting factor. As a result the estimate of $\boldsymbol{d}(k)$ converges to its true value in a short time. Unlike the case of constant bias, the time of failure occurrence as well as the extent of the failure must be estimated in a system with component failures.

4.1. *Corrected estimate of $\boldsymbol{x}$*

Assume that the magnitude $\boldsymbol{d}(k)$ and occurrence time $\theta(\leq k)$ of the failure are known; then the estimation results of the filter based on no failure assumption can be corrected by using the following equations (Friedland 1969, 1978).

$$\hat{\boldsymbol{x}}(k|k-1;\theta) = \tilde{\boldsymbol{x}}(k|k-1) + \mathbf{U}(k;\theta)\boldsymbol{d}(k-1;\theta) \tag{33}$$

$$\hat{\boldsymbol{x}}(k;\theta) = \tilde{\boldsymbol{x}}(k) + \mathbf{V}(k;\theta)\boldsymbol{d}(k;\theta) \tag{34}$$

where $\tilde{\boldsymbol{x}}$ is the estimation results from the filter based on no failure assumption, given in (3)–(6), and $\hat{\boldsymbol{x}}$ is the corrected state estimates.

The corrected estimate covariances are described by

$$\mathbf{P}(k|k-1;\theta) = \tilde{\mathbf{P}}(k|k-1) + \mathbf{U}(k;\theta)\mathbf{P}_d(k|k-1;\theta)\mathbf{U}^{\mathrm{T}}(k;\theta) \tag{35}$$

$$\mathbf{P}(k;\theta) = \tilde{\mathbf{P}}(k) + \mathbf{V}(k;\theta)\mathbf{P}_d(k;\theta)\mathbf{V}^{\mathrm{T}}(k;\theta) \tag{36}$$

where $\mathbf{P}_d$ is the estimate covariance of the failure state $\boldsymbol{d}$. $\mathbf{U}$ and $\mathbf{V}$, which are referred to as sensitivity matrices, are defined as

$$\left.\begin{aligned} \mathbf{V}(k+1;\theta) &= \mathbf{U}(k;\theta) - \mathbf{K}(k)\mathbf{S}(k;\theta), \quad \mathbf{V}(0;0) = 0 \\ \mathbf{U}(k;\theta) &= \mathbf{A}(k-1)\mathbf{V}(k;\theta) + \mathbf{E}_1(k-1) \\ \mathbf{S}(k;\theta) &= \mathbf{H}(k)\mathbf{U}(k;\theta) + \mathbf{E}_2(k) \end{aligned}\right\} \tag{37}$$

4.2. *Estimation of failure state*

Modifying the results in Friedland (1980) by using failure occurrence time θ, it can be shown that the estimation of failure state $\boldsymbol{d}(k;\theta)$ is equivalent to the estimation of a state vector with equivalent dynamic equation

$$\boldsymbol{d}(k+1;\theta) = \boldsymbol{d}(k;\theta) + \boldsymbol{w}_d(k) \tag{38}$$

and equivalent measurement equation

$$\tilde{\gamma}(k) = \mathbf{S}(k;\theta)\boldsymbol{d}(k;\theta) + \boldsymbol{v}_d(k) \tag{39}$$

where the matrix $\mathbf{S}(k;\theta)$ is an equivalent measurement matrix defined in (37), $\boldsymbol{w}_d(k)$ is white noise with covariance matrix $\mathbf{Q}_d(k)$ introduced to represent the model errors, and $\boldsymbol{v}_d(k)$ is white noise with an equivalent covariance matrix

$$\mathbf{R}_d(k) = \mathbf{H}(k)\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k) \tag{40}$$

Since the failure state described by (38) and (39) is subjected to a sudden change when some component has failed, it is estimated by using the adaptive fading Kalman filter

$$\hat{\boldsymbol{d}}(k;\theta) = \hat{\boldsymbol{d}}(k-1;\theta) + \mathbf{K}_d(k;\theta)\gamma_d(k;\theta) \tag{41}$$

that the gain matrix is given by

$$\left.\begin{aligned} \mathbf{K}_d(k;\theta) &= \mathbf{P}_d(k|k-1;\theta)\mathbf{S}^{\mathrm{T}}(k;\theta)[\mathbf{S}(k;\theta)\mathbf{P}_d(k|k-1;\theta)\mathbf{S}^{\mathrm{T}}(k;\theta) \\ &\quad + \mathbf{H}(k)\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k)]^{-1} \\ \mathbf{P}_d(k|k-1;\theta) &= \lambda_d(k;\theta)\mathbf{P}_d(k-1;\theta) + \mathbf{Q}_d(k-1) \\ \mathbf{P}_d(k;\theta) &= [\mathbf{I} - \mathbf{K}_d(k;\theta)\mathbf{S}(k;\theta)]\mathbf{P}_d(k|k-1;\theta) \end{aligned}\right\} \tag{42}$$

where $\gamma_d(k;\theta)$ is a new residual sequence given by

$$\gamma_d(k;\theta) = \tilde{\gamma}(k) - \mathbf{S}(k;\theta)\hat{\boldsymbol{d}}(k-1;\theta) \tag{43}$$

The key problem here is how to generate $\lambda_d(k;\theta)$. Comparing (38) and (39) with (1) and (2), the following propositions are obvious from Propositions 1–4, to generate $\lambda_d(k;\theta)$.

Proposition 5

Given failure state equations (38) and (39) with the two conditions of Proposition 1 satisfied: $\mathbf{Q}_d(k)$, $\mathbf{R}_d(k)$ and $\mathbf{P}_d(0)$ are positive definite for all k; and the measurement matrix $\mathbf{S}(k;\theta)$ is fully ranked for all k and θ. Then the optimal forgetting factor is computed by

$$\lambda_d(k;\theta) = \max\left\{1, \frac{1}{m}\operatorname{trace}[\mathbf{N}_d(k;\theta)\mathbf{M}_d^{-1}(k;\theta)]\right\} \tag{44}$$

where

$$\mathbf{M}_d(k;\theta) = \mathbf{S}(k;\theta)\mathbf{P}_d(k-1;\theta)\mathbf{S}^{\mathrm{T}}(k;\theta) \tag{45}$$

$$\begin{aligned} \mathbf{N}_d(k;\theta) &= \mathbf{C}_{d0}(k;\theta) - \mathbf{S}(k;\theta)\mathbf{Q}_d(k-1)\mathbf{S}^{\mathrm{T}}(k;\theta) \\ &\quad - \mathbf{H}(k)\tilde{\mathbf{P}}(k|k-1)\mathbf{H}^{\mathrm{T}}(k) - \mathbf{R}(k) \end{aligned} \tag{46}$$

Proposition 6

Failure state equations and conditions are the same as in Proposition 5. The optimal forgetting factor can be computed from

$$\lambda(k;\theta) = \max\{1, \operatorname{trace}[\mathbf{N}_d(k;\theta)]/\operatorname{trace}[\mathbf{M}_d(k;\theta)]\} \tag{47}$$

where the matrices $\mathbf{M}_d(k;\theta)$ and $\mathbf{N}_d(k;\theta)$ are defined by (45) and (46).

Proposition 7

Given failure state equations (38) and (39), the optimal forgetting factor $\lambda_d(k;\theta)$ can be obtained by iterative computation

$$\lambda_d^{l+1}(k;\theta) = \lambda_d^l(k;\theta) - \phi\frac{\partial f_d^l(k;\theta)}{\partial\lambda_d^l(k;\theta)} \tag{48}$$

with initial conditions

$$\lambda_d^0(0;0) = 1, \quad \lambda_d^0(k;\theta) = \lambda_d(k-1;\theta) \tag{49}$$

where the superscript l denotes the number of iterations at a time instant; ϕ is the step length. If the following condition holds at the pth iteration

$$|\lambda_d^p(k;\theta) - \lambda_d^{p-1}(k;\theta)| < \epsilon \tag{50}$$

then stop the iteration and let

$$\lambda_d(k;\theta) = \lambda_d^p(k;\theta) \tag{51}$$

The gradient term in (48) is

$$\frac{\partial f^l(k;\theta)}{\partial \lambda_d^l(k;\theta)} = \sum_{i=1}^{n}\sum_{j=1}^{m}\left[\frac{\partial \mathbf{Z}_d^l(k;\theta)}{\partial \lambda_d^l(k;\theta)}\right]_{ij} \tag{52}$$

$$\mathbf{Z}_d^l(k;\theta) = \mathbf{P}_d^l(k|k-1;\theta)\mathbf{S}^{\mathrm{T}}(k;\theta) - \mathbf{K}_d^l(k;\theta)\mathbf{C}_{d0}(k;\theta) \tag{53}$$

In Propositions 5–7 $\mathbf{C}_{d0}(k;\theta)$ is the error covariance of failure state estimate defined by

$$\mathbf{C}_{d0}(k;\theta) = \mathbf{E}[\gamma_d(k;\theta)\gamma_d^{\mathrm{T}}(k;\theta)] \tag{54}$$

which can be calculated from the following proposition.

Proposition 8

The covariance of $\gamma_d(k;\theta)$ can be estimated by a recursive fading formula

$$\mathbf{C}_{d0}(k;\theta) = \mathbf{G}_{d1}(k;\theta)/\mathbf{G}_{d2}(k;\theta) \tag{55}$$

$$\mathbf{G}_{d1}(k;\theta) = \mathbf{G}_{d1}(k-1;\theta)/\lambda_d(k-1;\theta) + \gamma_d(k;\theta)\gamma_d^{\mathrm{T}}(k;\theta) \tag{56}$$

$$\mathbf{G}_{d2}(k;\theta) = \mathbf{G}_{d2}(k-1;\theta)/\lambda_d(k-1;\theta) + 1, \quad \mathbf{G}_{d1}(0;0) = 0, \quad \mathbf{G}_{d2}(0;0) = 0 \tag{57}$$

4.3. *Estimation of failure time θ*

The estimation of failure occurrence time θ is obtained by calculating the probability of each failure time hypothesis. Let $p(k;\theta)$ denote the probability that failure occurs at time θ based on measured data through the kth measurement, i.e. given $\mathbf{I}_k = \{\boldsymbol{u}(0),\ldots,\boldsymbol{u}(k-1), \boldsymbol{y}(1),\ldots,\boldsymbol{y}(k)\}$, then Bayes' rule yields the following recursive formula for the $p(k;\theta)$.

$$p(k+1;\theta) = \frac{p(y(k+1)|\theta, \mathbf{I}_k, \boldsymbol{u}(k))p(k;\theta)}{\sum_{t=0}^{k} p(y(k+1)|t, \mathbf{I}_k, \boldsymbol{u}(k)))p(k;t)}, \quad \theta = 1, 2, \ldots, k \tag{58}$$

By defining the corrected residual as

$$\gamma(k;\theta) = \boldsymbol{y}(k) - \mathbf{H}(k)\hat{\boldsymbol{x}}(k|k-1;\theta) - \mathbf{E}_2(k)\hat{\boldsymbol{d}}(k-1;\theta) \tag{59}$$

where $\gamma(k;\theta)$ is zero mean white noise with covariance $\mathbf{V}(k;\theta)$, while θ is the true failure occurrence time

$$\mathbf{V}(k;\theta) = \mathbf{H}(k)\mathbf{P}(k|k-1;\theta)\mathbf{H}^{\mathrm{T}}(k) + \mathbf{R}(k) \tag{60}$$

then

$$p(y(k)|\theta, \mathbf{I}_{k-1}, \boldsymbol{u}(k-1)) = \frac{\exp\left[\frac{1}{2}\gamma(k;\theta)\mathbf{V}^{-1}(k;\theta)\gamma^{\mathrm{T}}(k;\theta)\right]}{(2\pi)^{m/2}[\det \mathbf{V}(k;\theta)]^{1/2}} \tag{61}$$

Equation (58) involves a growing bank of failure state estimators. To avoid this problem, we constrain the failure time to an interval of the form $k - \mathbf{M} \le \theta \le k - \mathbf{N}$, referred to as a sliding window. It is very important to select a window of appropriate width so that both the performance requirement and the computational burden limit are satisfied.

A convenient and efficient way to reduce the width of the sliding window is by making an extra failure time hypothesis $\theta = \theta_0 < k - \mathbf{M}$, besides the window. By defining $\mathbf{J}_k = \{\theta_0, k - \mathbf{M}, k - \mathbf{M} + 1, \ldots, k - \mathbf{N}\}$, the recursive formula for the $\boldsymbol{p}(k;\theta)$ is

$$\boldsymbol{p}(k+1;\theta) = \frac{\boldsymbol{p}(y(k+1)|\theta, \mathbf{I}_k, \boldsymbol{u}(k))\boldsymbol{p}(k;\theta)}{\sum_{t \in J_k} \boldsymbol{p}(\boldsymbol{y}(k+1))|t, \mathbf{I}_k, \boldsymbol{u}(k))\boldsymbol{p}(k;t)} \tag{62}$$

At each instant k, compare magnitudes of $\boldsymbol{p}(k;t)$ and $\boldsymbol{p}(k;\theta_0)$. If

$$\boldsymbol{p}(k;\theta_0) \ge \boldsymbol{p}(k;t), \quad \forall k - \mathbf{M} \le t \le k - \mathbf{N} \tag{63}$$

then θ_0 is still the failure time estimate for the next instant, otherwise

$$\theta_0 = k - M \tag{64}$$

In this way, no matter how narrow the window is, the true failure time will remain in $\mathbf{J}_k$. This method is also applicable to model error estimation.

From (34), noting that

$$\sum_{\theta \in J_k} \boldsymbol{p}(k;\theta) = 1 \tag{65}$$

the final corrected least mean square estimate of the state is given by

$$\hat{\boldsymbol{x}}(k) = \tilde{\boldsymbol{x}}(k) + \sum_{\theta \in J_k} \boldsymbol{p}(k;\theta)\mathbf{V}(k;\theta)\hat{\boldsymbol{d}}(k;\theta) \tag{66}$$

4.4. *Stepwise compensation of failure state*

The technique developed is efficient in the case where all failures occur at the same moment. If different failures occur at different moments, the technique will give wrong detection results. To solve this problem, we can simply compensate the failure state in the filter based on the no failure assumption in every prescribed time interval.

The equations which describe the filter based on the no failure assumption given in (3)–(5) are rewritten as

$$\tilde{\boldsymbol{x}}(k|k-1) = \mathbf{A}(k-1)\tilde{\boldsymbol{x}}(k-1) + \mathbf{B}(k)\boldsymbol{u}(k-1) + \mathbf{E}_1(k-1)\boldsymbol{d}_0(k-1) \tag{67}$$

$$\tilde{\boldsymbol{x}}(k) = \tilde{\boldsymbol{x}}(k|k-1) + \mathbf{K}(k)\tilde{\gamma}(k) \tag{68}$$

$$\tilde{\gamma}(k) = \boldsymbol{y}(k) - \mathbf{H}(k)\tilde{\boldsymbol{x}}(k|k-1) - \mathbf{E}_2(k)\boldsymbol{d}_0(k) \tag{69}$$

where $\boldsymbol{d}_0(k)$ is a stepwisely changed known vector with $\boldsymbol{d}_0(0) = 0$.

When a failure occurs and the estimate converge to a new value, or in every prespecified time interval, adjust $\boldsymbol{d}_0(k)$. By denoting the instant to compensate the failure state by k_0, then

$$\boldsymbol{d}_0(k_0) = \boldsymbol{d}_0(k_0) + \hat{\boldsymbol{d}}(k_0) \tag{70}$$

where

$$\hat{\boldsymbol{d}}(k_0) = \sum_{\theta \in J_k} \boldsymbol{p}(k_0;\theta)\hat{\boldsymbol{d}}(k_0;\theta) \tag{71}$$

At the meantime, the filter based on no failure assumption is reinitialized

$$\tilde{x}(k_0) = \hat{x}(k_0) \tag{72}$$

$$\tilde{\mathbf{P}}(k_0) = \tilde{\mathbf{P}}(k_0) + \sum_{\theta \in J_k} p(k_0; \theta)\mathbf{V}(k_0; \theta)\mathbf{P}_d(k_0; \theta)\mathbf{V}^{\mathrm{T}}(k_0; \theta) \tag{73}$$

$$\hat{d}(k_0; \theta) = 0 \tag{74}$$

In (70) and (73), $d_0(k_0)$ and $\tilde{\mathbf{P}}(k_0)$ on the right-hand side are the variables before reinitialization, and those on the left-hand side are the variables after reinitialization. The detection of new failure begins after the reinitialization procedures.

The stepwise compensation method makes it possible to discriminate between model errors and component failures. If the estimate of failure state $\hat{d}(k)$ computed from (71) changes very slowly in a period of L sampling times, it means that no failure occurs. In this case $\hat{d}(k)$ is the best approximation of the model errors represented by $d(k)$. If $\hat{d}(k)$ changes quickly from one value to another, it can be assumed that some components have failed. In this way the effect of model errors on the detection of component failure is reduced to a very low level.

The stepwise compensation method can also remove the accumulated effects of model errors and component failures in the filter. As a result, the divergence problem of the filter based on no failure assumption is avoided.

5. Conclusion

In this paper we have described a failure detection technique for linear systems. The technique can simultaneously detect, estimate and compensate the actuator and sensor failures, as well as model errors, with good accuracy and rapid response. Because of its modest computation burden and robustness to the model errors, the technique is especially applicable to on-line process control systems. Although we focused attention here on the actuator and sensor failure detection problem, the technique can be also effectively applied to the general system failure detection problem.

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