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SHORT COMMUNICATION

DECOMPOSITION METHOD FOR SOLVING KALMAN FILTER GAINS IN SINGULARLY PERTURBED SYSTEMS

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SUMMARY

In this paper a decomposition method is introduced for the solution of the optimal Kalman filter gains in singularly perturbed systems by solving two reduced-order linear equations. The decomposition is achieved via the use of the Chang transformation, which is applied to the Hamiltonian matrix of the singularity perturbed Kalman filter. Since the decoupling transformation can be obtained up to an arbitrary degree of accuracy at very low cost, this approach produces an efficient numerical method for obtaining the Kalman filter gains. A numerical example is given to demonstrate the efficiency of the method.

KEY WORDS Kalman filter Singular perturbation Decomposition Riccati equation

1. INTRODUCTION

Linear singularly perturbed systems have been studied recently in different set-ups by many researchers.[1–7] The gain determination problem for a Kalman filter in a singularly perturbed system has been considered using the numerical power series expansion method.[4–7] The approximate solution for the Kalman filter gain is obtained in terms of an outer series and a correction series.[6,7] The outer series takes advantage of the order reduction associated with degeneration (the degenerate equations obtained from the suppression of a small parameter violate some of the given initial conditions) and the correction series takes care of the violated initial conditions. The steady state Kalman filter gain is obtained by solving the reduced-order linear algebraic equations instead of the non-linear coupled algebraic equations.[4–6] Both solutions have accuracy $O(\varepsilon)$, where ε is small perturbation parameter. Being non-recursive in nature, the power series expansion method becomes very cumbersome and computationally very expensive when a high order of accuracy is required.

In this paper we will present an efficient recursive numerical method for the solution of the Kalman filter gain in a singularly perturbed system. The proposed method produces to a near-optimal solution with an arbitrary order of accuracy, i.e. $O(\varepsilon^i)$.

Consider the design of a Kalman filter for large-scale linear singularly perturbed systems of the form[4–7]

$$\dot{\mathbf{x}}_1(t) = \mathbf{A}_1\mathbf{x}_1(t) + \mathbf{A}_2\mathbf{x}_2(t) + \mathbf{G}_1\mathbf{w}(t), \qquad \mathbf{x}_1(0) = \mathbf{x}_{10} \tag{1}$$

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$$\varepsilon\dot{\mathbf{x}}_2(t) = \mathbf{A}_3\mathbf{x}_1(t) + \mathbf{A}_4\mathbf{x}_2(t) + \mathbf{G}_2\mathbf{w}(t), \qquad \mathbf{x}_2(0) = \mathbf{x}_{20} \tag{2}$$

$$\mathbf{y}(t) = \mathbf{C}_1\mathbf{x}_1(t) + \mathbf{C}_2\mathbf{x}_2(t) + \mathbf{v}(t) \tag{3}$$

with slow states $\mathbf{x}_1(t) \in \mathbb{R}^{n_1}$, fast states $\mathbf{x}_2(t) \in \mathbb{R}^{n_2}$ and measurements $\mathbf{y}(t) \in \mathbb{R}^{l}$; $\mathbf{w}(t) \in \mathbb{R}^{s}$ and $\mathbf{v}(t) \in \mathbb{R}^{l}$ are independent zero-mean stationary white Gaussian noise processes with intensities $\mathbf{W} > 0$ and $\mathbf{V} > 0$. The matrices $\mathbf{W}$ and $\mathbf{V}$ will often be referred to as the noise covariance matrices.

For the system (1)–(3) the Kalman filter becomes

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{K}(t)[\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)] \tag{4}$$

where $\mathbf{K}(t)$ is the gain of the Kalman filter given by

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1} \tag{5}$$

and $\mathbf{P}(t)$ satisfies the differential Riccati equation

$$\dot{\mathbf{P}}(t) = \mathbf{P}(t)\mathbf{A}^{\mathrm{T}} + \mathbf{A}\mathbf{P}(t) - \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{P}(t) + \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}, \qquad \mathbf{P}(t_0) = \mathbf{P}_0 \tag{6}$$

where

$$\hat{\mathbf{x}}(t) = \begin{pmatrix}\hat{\mathbf{x}}_1(t)\\ \hat{\mathbf{x}}_2(t)\end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix}\mathbf{A}_1 & \mathbf{A}_2\\ \mathbf{A}_3/\varepsilon & \mathbf{A}_4/\varepsilon\end{pmatrix}, \qquad \mathbf{C} = (\mathbf{C}_1 \ \ \mathbf{C}_2), \qquad \mathbf{G} = \begin{pmatrix}\mathbf{G}_1\\ \mathbf{G}_2/\varepsilon\end{pmatrix}$$

The presence of the small parameter ε makes this problem numerically ill-defined, producing a so-called stiff numerical problem. In order to overcome this difficulty and obtain an efficient numerical method for solving (6), we will utilize the known Hamiltonian form of the solution of the Riccati equation[8] and a non-singular Chang transformation.[9] The Hamiltonian form can 'linearize' the differential Riccati equation and the Chang transformation is used to block diagonalize the Hamiltonian, so that the required solution of the Riccati equation is obtained in terms of reduced-order problems. In addition, an efficient Newton-type algorithm[10] (with quadratic rate of convergence, i.e. $O(\varepsilon^{2^i})$, where i is the number of iterations) is used to solve the algebraic equation, which results in forming the Chang transformation. A numerical example is given to demonstrate the efficiency of the proposed method.

2. MAIN RESULTS

Consider the pair of linear matrix differential equations[11,12]

$$\dot{\mathbf{M}}(t) = \mathbf{A}\mathbf{M}(t) + \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}\mathbf{N}(t), \qquad \mathbf{M}(t_0) = \mathbf{P}_0 \tag{7}$$

$$\dot{\mathbf{N}}(t) = \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{M}(t) - \mathbf{A}^{\mathrm{T}}\mathbf{N}(t), \qquad \mathbf{N}(t_0) = \mathbf{I} \tag{8}$$

It can be shown by direct substitution that

$$\mathbf{M}(t) = \mathbf{P}(t)\mathbf{N}(t) \tag{9}$$

Differentiating this form gives

$$\dot{\mathbf{M}}(t) = \dot{\mathbf{P}}(t)\mathbf{N}(t) + \mathbf{P}(t)\dot{\mathbf{N}}(t) \tag{10}$$

Using equations (4)–(6), this becomes

$$\mathbf{A}\mathbf{M}(t) + \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}\mathbf{N}(t) = \dot{\mathbf{P}}(t)\mathbf{N}(t) + \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{M}(t) - \mathbf{P}(t)\mathbf{A}^{\mathrm{T}}\mathbf{N}(t) \tag{11}$$

Substituting (9) and collecting terms produces the equation

$$[\dot{\mathbf{P}}(t) - \mathbf{P}(t)\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{P}(t) - \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}]\mathbf{N}(t) = 0 \tag{12}$$

Now, if $N(t)$ is non-singular for all $t \geqslant t_0$, this equation is equivalent to (6) and

$$\mathbf{P}(t) = \mathbf{M}(t)\mathbf{N}^{-1}(t) \tag{13}$$

Note also that

$$\mathbf{P}(t_0) = \mathbf{M}(t_0)\mathbf{N}^{-1}(t_0) = \mathbf{P}_0\mathbf{I} = \mathbf{P}_0 \tag{14}$$

Thus the initial condition is also satisfied.

To show that $\mathbf{N}(t)$ is non-singular, it is noted that

$$\dot{\mathbf{N}}(t) = \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{M}(t) - \mathbf{A}^{\mathrm{T}}\mathbf{N}(t) = [-\mathbf{A}^{\mathrm{T}} + \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C}\mathbf{P}(t)]\,\mathbf{N}(t) \tag{15}$$

Since $\mathbf{N}(t_0) = \mathbf{I}$, $\mathbf{N}(t)$ is a transition matrix and hence non-singular.

We know the nature of the solution of (6), which is properly scaled as[1,10]

$$\mathbf{P}(t) = \begin{pmatrix} \mathbf{P}_1(t) & \mathbf{P}_2(t) \\ \mathbf{P}_2^T(t) & P_3/\varepsilon(t) \end{pmatrix}, \qquad \mathbf{P}(t_0) = \begin{pmatrix} \mathbf{P}_1(t_0) & \mathbf{P}_2(t_0) \\ \mathbf{P}_2^T(t_0) & \mathbf{P}_3/\varepsilon(t_0) \end{pmatrix} \tag{16}$$

where $\dim \mathbf{P}_1 = \mathbf{n}_1 \times n_1$ and $\dim \mathbf{P}_3 = n_2 \times n_2$, with $n_1 + n_2 = n$ (n_1, slow variables; n_2, fast variables[1]).

We introduce compatible partitions of the matrices $\mathbf{M}(t)$ and $\mathbf{N}(t)$ as

$$\mathbf{M}(t) = \begin{pmatrix} \mathbf{M}_1(t) & \mathbf{M}_2(t) \\ \mathbf{M}_3(t) & \mathbf{M}_4(t) \end{pmatrix}, \qquad \mathbf{N}(t) = \begin{pmatrix} \mathbf{N}_1(t) & \mathbf{N}_2(t) \\ \varepsilon\mathbf{N}_3(t) & \varepsilon\mathbf{N}_4(t) \end{pmatrix} \tag{17}$$

Partitioning (7) and (8) according to (17) will reveal a decoupled structure, i.e. the equations for $\mathbf{M}_1(t)$, $\mathbf{M}_3(t)$, $\mathbf{N}_1(t)$ and $\mathbf{N}_3(t)$ are independent of the equations for $\mathbf{M}_2(t)$, $\mathbf{M}_4(t)$, $\mathbf{N}_2(t)$ and $\mathbf{N}_4(t)$ and vice versa:

$$\begin{pmatrix} \dot{\mathbf{M}}_1(t) \\ \dot{\mathbf{M}}_3(t) \\ \dot{\mathbf{N}}_1(t) \\ \varepsilon\dot{\mathbf{N}}_3(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{Q}_1 & \mathbf{Q}_2/\varepsilon \\ \mathbf{A}_3/\varepsilon & \mathbf{A}_4/\varepsilon & \mathbf{Q}_2^T/\varepsilon & \mathbf{Q}_3/\varepsilon^2 \\ \mathbf{S}_1 & \mathbf{S}_2 & -\mathbf{A}_1^T & -\mathbf{A}_3^T/\varepsilon \\ \mathbf{S}_2^T & \mathbf{S}_3 & -\mathbf{A}_2^T & -\mathbf{A}_4^T/\varepsilon \end{pmatrix} \begin{pmatrix} \mathbf{M}_1(t) \\ \mathbf{M}_3(t) \\ \mathbf{N}_1(t) \\ \varepsilon\mathbf{N}_3(t) \end{pmatrix} = \mathscr{H} \begin{pmatrix} \mathbf{M}_1(t) \\ \mathbf{M}_3(t) \\ \mathbf{N}_1(t) \\ \varepsilon\mathbf{N}_3(t) \end{pmatrix} \tag{18}$$

$$\begin{pmatrix} \dot{\mathbf{M}}_2(t) \\ \dot{\mathbf{M}}_4(t) \\ \dot{\mathbf{N}}_2(t) \\ \varepsilon\dot{\mathbf{N}}_4(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{Q}_1 & \mathbf{Q}_2/\varepsilon \\ \mathbf{A}_3/\varepsilon & \mathbf{A}_4/\varepsilon & \mathbf{Q}_2^T/\varepsilon & \mathbf{Q}_3/\varepsilon^2 \\ \mathbf{S}_1 & \mathbf{S}_2 & -\mathbf{A}_1^T & -\mathbf{A}_3^T/\varepsilon \\ \mathbf{S}_2^T & \mathbf{S}_3 & -\mathbf{A}_2^T & -\mathbf{A}_4^T/\varepsilon \end{pmatrix} \begin{pmatrix} \mathbf{M}_2(t) \\ \mathbf{M}_4(t) \\ \mathbf{N}_2(t) \\ \varepsilon\mathbf{N}_4(t) \end{pmatrix} = \mathscr{H} \begin{pmatrix} \mathbf{M}_2(t) \\ \mathbf{M}_4(t) \\ \mathbf{N}_2(t) \\ \varepsilon\mathbf{N}_4(t) \end{pmatrix} \tag{19}$$

where

$$\mathbf{Q} = \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}} = \begin{pmatrix} \mathbf{G}_1\mathbf{W}\mathbf{G}_1^{\mathrm{T}} & \mathbf{G}_1\mathbf{W}\mathbf{G}_2^{\mathrm{T}}/\varepsilon \\ \mathbf{G}_2\mathbf{W}\mathbf{G}_1^{\mathrm{T}}/\varepsilon & \mathbf{G}_2\mathbf{W}\mathbf{G}_2^{\mathrm{T}}/\varepsilon^2 \end{pmatrix} = \begin{pmatrix} \mathbf{Q}_1 & \mathbf{Q}_2/\varepsilon \\ \mathbf{Q}_2^{\mathrm{T}}/\varepsilon & \mathbf{Q}_3/\varepsilon^2 \end{pmatrix}$$

$$\mathbf{S} = \mathbf{C}^{\mathrm{T}}\mathbf{V}\mathbf{C} = \begin{pmatrix} \mathbf{C}_1^{\mathrm{T}}\mathbf{V}\mathbf{C}_1 & \mathbf{C}_1^{\mathrm{T}}\mathbf{V}\mathbf{C}_2 \\ \mathbf{C}_2^{\mathrm{T}}\mathbf{V}\mathbf{C}_1 & \mathbf{C}_2^{\mathrm{T}}\mathbf{V}\mathbf{C}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_2^{\mathrm{T}} & \mathbf{S}_3 \end{pmatrix}$$

Interchanging the second and third rows in (18) and (19) respectively produces

$$\begin{pmatrix} \dot{\mathbf{M}}_1(t) \\ \dot{\mathbf{N}}_1(t) \\ \varepsilon\dot{\mathbf{M}}_3(t) \\ \varepsilon\dot{\mathbf{N}}_3(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{Q}_1 & \mathbf{A}_2 & \mathbf{Q}_2 \\ \mathbf{S}_1 & -\mathbf{A}_1^{\mathrm{T}} & \mathbf{S}_2 & -\mathbf{A}_3^{\mathrm{T}} \\ \mathbf{A}_3 & \mathbf{Q}_2^{\mathrm{T}} & \mathbf{A}_4 & \mathbf{Q}_3 \\ \mathbf{S}_2^{\mathrm{T}} & -\mathbf{A}_2^{\mathrm{T}} & \mathbf{S}_3 & -\mathbf{A}_4^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \mathbf{M}_1(t) \\ \mathbf{N}_1(t) \\ \mathbf{M}_3(t) \\ \mathbf{N}_3(t) \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{pmatrix} \begin{pmatrix} \mathbf{M}_1(t) \\ \mathbf{N}_1(t) \\ \mathbf{M}_3(t) \\ \mathbf{N}_3(t) \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} \dot{\mathbf{M}}_2(t) \\ \dot{\mathbf{N}}_2(t) \\ \varepsilon\dot{\mathbf{M}}_4(t) \\ \varepsilon\dot{\mathbf{N}}_4(t) \end{pmatrix} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{Q}_1 & \mathbf{A}_2 & \mathbf{Q}_2 \\ \mathbf{S}_1 & -\mathbf{A}_1^T & \mathbf{S}_2 & -\mathbf{A}_3^T \\ \mathbf{A}_3 & \mathbf{Q}_2^T & \mathbf{A}_4 & \mathbf{Q}_3 \\ \mathbf{S}_2^T & -\mathbf{A}_2^T & \mathbf{S}_3 & -\mathbf{A}_4^T \end{pmatrix} \begin{pmatrix} \mathbf{M}_2(t) \\ \mathbf{N}_2(t) \\ \mathbf{M}_4(t) \\ \mathbf{N}_4(t) \end{pmatrix} = \begin{pmatrix} \mathbf{T}_1 & \mathbf{T}_2 \\ \mathbf{T}_3 & \mathbf{T}_4 \end{pmatrix} \begin{pmatrix} \mathbf{M}_2(t) \\ \mathbf{N}_2(t) \\ \mathbf{M}_4(t) \\ \mathbf{N}_4(t) \end{pmatrix} \tag{21}$$

where

$$\mathbf{T}_1 = \begin{pmatrix} \mathbf{A}_1 & \mathbf{Q}_1 \\ \mathbf{S}_1 & -\mathbf{A}_1^T \end{pmatrix}, \qquad \mathbf{T}_2 = \begin{pmatrix} \mathbf{A}_2 & \mathbf{Q}_2 \\ \mathbf{S}_2 & -\mathbf{A}_3^T \end{pmatrix}, \qquad \mathbf{T}_3 = \begin{pmatrix} \mathbf{A}_3 & \mathbf{Q}_2^T \\ \mathbf{S}_2^T & -\mathbf{A}_2^T \end{pmatrix},$$

$$\mathbf{T}_4 = \begin{pmatrix} \mathbf{A}_4 & \mathbf{Q}_3 \\ \mathbf{S}_3 & -\mathbf{A}_4^T \end{pmatrix}$$

Introducing the notations

$$\mathbf{U}(t) = \begin{pmatrix} \mathbf{M}_1(t) \\ \mathbf{N}_1(t) \end{pmatrix} = \begin{pmatrix} \mathbf{U}_1(t) \\ \mathbf{U}_2(t) \end{pmatrix}, \qquad \mathbf{Z}(t) = \begin{pmatrix} \mathbf{M}_3(t) \\ \mathbf{N}_3(t) \end{pmatrix} = \begin{pmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{pmatrix} \tag{22}$$

$$\mathbf{X}(t) = \begin{pmatrix} \mathbf{M}_2(t) \\ \mathbf{N}_2(t) \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1(t) \\ \mathbf{X}_2(t) \end{pmatrix}, \qquad \mathbf{Y}(t) = \begin{pmatrix} \mathbf{M}_4(t) \\ \mathbf{N}_4(t) \end{pmatrix} = \begin{pmatrix} \mathbf{Y}_1(t) \\ \mathbf{Y}_2(t) \end{pmatrix} \tag{23}$$

we get two systems of singularly perturbed matrix equations

$$\dot{\mathbf{U}}(t) = \mathbf{T}_1\mathbf{U}(t) + \mathbf{T}_2\mathbf{Z}(t), \qquad \varepsilon\dot{\mathbf{Z}}(t) = \mathbf{T}_3\mathbf{U}(t) + \mathbf{T}_4\mathbf{Z}(t) \tag{24}$$

$$\dot{\mathbf{X}}(t) = \mathbf{T}_1\mathbf{X}(t) + \mathbf{T}_2\mathbf{Y}(t), \qquad \varepsilon\dot{\mathbf{Y}}(t) = \mathbf{T}_3\mathbf{X}(t) + \mathbf{T}_4\mathbf{Y}(t) \tag{25}$$

with new initial conditions

$$\mathbf{U}(t_0) = \begin{pmatrix} \mathbf{P}_1(0) \\ \mathbf{I}_{n_1} \end{pmatrix}, \qquad \mathbf{Z}(t_0) = \begin{pmatrix} \mathbf{P}_2^T(0) \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{X}(t_0) = \begin{pmatrix} \mathbf{P}_2(0) \\ \mathbf{0} \end{pmatrix}, \qquad \mathbf{Y}(t_0) = \begin{pmatrix} \mathbf{P}_3(0)/\varepsilon \\ \mathbf{I}_{n_2}/\varepsilon \end{pmatrix}$$

Note that the systems (24) and (25) have exactly the same form, the only difference being the initial conditions.

In the sequel we introduce the transformation[9] defined by

$$\mathbf{J} = \begin{pmatrix} \mathbf{I}_{2n_1} - \varepsilon\mathbf{HL} & -\varepsilon\mathbf{H} \\ \mathbf{L} & \mathbf{I}_{2n_2} \end{pmatrix}, \qquad \mathbf{J}^{-1} = \begin{pmatrix} \mathbf{I}_{2n_1} & \varepsilon\mathbf{H} \\ -\mathbf{L} & \mathbf{I}_{2n_2} - \varepsilon\mathbf{LH} \end{pmatrix} \tag{26}$$

where $\mathbf{L}$ and $\mathbf{H}$ satisfy

$$\mathbf{T}_4\mathbf{L} - \mathbf{T}_3 - \varepsilon\mathbf{L}(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L}) = 0 \tag{27}$$

$$-\mathbf{H}(\mathbf{T}_4 + \varepsilon\mathbf{L}\mathbf{T}_2) + \mathbf{T}_2 + \varepsilon(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})\mathbf{H} = 0 \tag{28}$$

The matrices $\mathbf{L}$ and $\mathbf{H}$ can be obtained by using a recursive Newton-type algorithm[10,13] with quadratic rate of convergence; the algorithm is briefly summarized in the following.

For the algebraic equation (27) the initial guess is easily obtained to $O(\varepsilon)$ accuracy by setting $\varepsilon = 0$ in the equation, i.e.

$$\mathbf{L}^{(0)} = \mathbf{T}_4^{-1}\mathbf{T}_3 = \mathbf{L} + O(\varepsilon) \tag{29}$$

Thus the Newton sequence will be $O(\varepsilon^2)$, $O(\varepsilon^4)$, $O(\varepsilon^8)$, ..., $O(\varepsilon^{2^i})$, close to the exact solution respectively in each iteration.

The Newton-type algorithm of (27) can be constructed by setting $\mathbf{L}^{(i+1)} = \mathbf{L}^{(i)} + \Delta\mathbf{L}^{(i)}$ and neglecting $O((\Delta \mathbf{L})^2)$ terms. This will produce a Lyapunov-type equation of the form

$$\mathbf{D}_1^{(i)}\mathbf{L}^{(i+1)} + \mathbf{L}^{(i+1)}\mathbf{D}_2^{(i)} = \mathbf{Q}^{(i)} \tag{30}$$

where

$$\mathbf{D}_1^{(i)} = \mathbf{T}_4 + \varepsilon\mathbf{L}^{(i)}\mathbf{T}_2, \qquad \mathbf{D}_2^{(i)} = -\varepsilon(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L}^{(i)})$$
$$\mathbf{Q}^{(i)} = \mathbf{T}_3 + \varepsilon\mathbf{L}^{(i)}\mathbf{T}_2\mathbf{L}^{(i)}, \quad i = 0, 1, 2, \ldots$$

with the initial condition given by (29).

Having found the solution of (27) up to the required degree of accuracy, one can get the solution of (28) by solving directly a Lyapunov equation of the form

$$\mathbf{H}^{(i)}\mathbf{D}_1^{(i)} + \mathbf{D}_2^{(i)}\mathbf{H}^{(i)} = \mathbf{T}_2 \tag{31}$$

which implies $\mathbf{H}^{(i)} = \mathbf{H} + O(\varepsilon^{2^i})$.

The transformation (26) is then applied to (24) and (25) to give

$$\begin{pmatrix}\underline{\mathbf{U}}(t)\\ \underline{\mathbf{Z}}(t)\end{pmatrix} = \mathbf{J}\begin{pmatrix}\mathbf{U}(t)\\ \mathbf{Z}(t)\end{pmatrix}, \qquad \begin{pmatrix}\underline{\mathbf{X}}(t)\\ \underline{\mathbf{Y}}(t)\end{pmatrix} = \mathbf{J}\begin{pmatrix}\mathbf{X}(t)\\ \mathbf{Y}(t)\end{pmatrix} \tag{32}$$

with

$$\begin{pmatrix}\underline{\mathbf{U}}(t_0)\\ \underline{\mathbf{Z}}(t_0)\end{pmatrix} = \mathbf{J}\begin{pmatrix}\mathbf{U}(t_0)\\ \mathbf{Z}(t_0)\end{pmatrix}, \qquad \begin{pmatrix}\underline{\mathbf{X}}(t_0)\\ \underline{\mathbf{Y}}(t_0)\end{pmatrix} = \mathbf{J}\begin{pmatrix}\mathbf{X}(t_0)\\ \mathbf{Y}(t_0)\end{pmatrix} \tag{32}$$

This will produce two completely decoupled subsystems

$$\dot{\underline{\mathbf{U}}} = (\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})\underline{\mathbf{U}}, \qquad \underline{\mathbf{U}}(t_0) = (\mathbf{I}_{2n_1} - \varepsilon\mathbf{HL})\mathbf{U}(t_0) - \varepsilon\mathbf{HZ}(t_0) \tag{33}$$

$$\varepsilon\dot{\underline{\mathbf{Z}}} = (\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)\underline{\mathbf{Z}}, \qquad \underline{\mathbf{Z}}(t_0) = \mathbf{LU}(t_0) + \mathbf{Z}(t_0) \tag{34}$$

and

$$\dot{\underline{\mathbf{X}}} = (\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})\underline{\mathbf{X}}, \qquad \underline{\mathbf{X}}(t_0) = (\mathbf{I}_{2n_1} - \varepsilon\mathbf{HL})\mathbf{X}(t_0) - \varepsilon\mathbf{HY}(t_0) \tag{35}$$

$$\varepsilon\dot{\underline{\mathbf{Y}}} = (\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)\underline{\mathbf{Y}}, \qquad \underline{\mathbf{Y}}(t_0) = \mathbf{LX}(t_0) + \mathbf{Y}(t_0) \tag{36}$$

The solutions of (33)–(36) are given by

$$\underline{\mathbf{U}}(t) = \mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{U}}(t_0), \qquad \underline{\mathbf{Z}}(t) = \mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Z}}(t_0) \tag{37}$$

$$\underline{\mathbf{X}}(t) = \mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{X}}(t_0), \qquad \underline{\mathbf{Y}}(t) = \mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Y}}(t_0) \tag{38}$$

From equation (32) we have

$$\begin{pmatrix}\mathbf{U}(t)\\ \mathbf{Z}(t)\end{pmatrix} = \mathbf{J}^{-1}\begin{pmatrix}\underline{\mathbf{U}}(t)\\ \underline{\mathbf{Z}}(t)\end{pmatrix} \qquad \begin{pmatrix}\mathbf{X}(t)\\ \mathbf{Y}(t)\end{pmatrix} = \mathbf{J}^{-1}\begin{pmatrix}\underline{\mathbf{X}}(t)\\ \underline{\mathbf{Y}}(t)\end{pmatrix} \tag{39}$$

Then the solutions in the original co-ordinates are

$$\mathbf{U}(t) = \mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{U}}(t_0) + \varepsilon\mathbf{H}\mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Z}}(t_0) \tag{40}$$

$$\mathbf{Z}(t) = -\mathbf{L}\mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{U}}(t_0) + (\mathbf{I} - \varepsilon\mathbf{LH})\mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Z}}(t_0) \tag{41}$$

and

$$\mathbf{X}(t) = \mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{X}}(t_0) + \varepsilon\mathbf{H}\mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Y}}(t_0) \tag{42}$$

$$\mathbf{Y}(t) = -\mathbf{L}\mathrm{e}^{(\mathbf{T}_1 - \mathbf{T}_2\mathbf{L})t}\underline{\mathbf{X}}(t_0) + (\mathbf{I} - \varepsilon\mathbf{LH})\mathrm{e}^{(1/\varepsilon)(\mathbf{T}_4 + \varepsilon\mathbf{LT}_2)t}\underline{\mathbf{Y}}(t_0) \tag{43}$$

Partitioning (40)–(43) according to (22) and (23) will produce all components of the matrices $\mathbf{M}(t)$ and $\mathbf{N}(t)$, so that the required solution of (6) is given by

$$\mathbf{P}(t)=\begin{pmatrix}\mathbf{M}_1(t) & \mathbf{M}_2(t)\\ \mathbf{M}_3(t) & \mathbf{M}_4(t)\end{pmatrix}\begin{pmatrix}\mathbf{N}_1(t) & \mathbf{N}_2(t)\\ \varepsilon\mathbf{N}_3(t) & \varepsilon\mathbf{N}_4(t)\end{pmatrix}^{-1}=\begin{pmatrix}\mathbf{U}_1(t) & \mathbf{X}_1(t)\\ \mathbf{Z}_1(t) & \mathbf{Y}_1(t)\end{pmatrix}\begin{pmatrix}\mathbf{U}_2(t) & \mathbf{X}_2(t)\\ \varepsilon\mathbf{Z}_2(t) & \varepsilon\mathbf{Y}_2(t)\end{pmatrix}^{-1} \tag{44}$$

Thus, in order to get the solution of (6), i.e. $\mathbf{P}(t)$, which has $\dim(n\times n)=\dim(n_1+n_2)\times(n_1+n_2)$, we only have to solve two simple algebraic equations, (27) of $\dim(2n_2\times 2n_1)$ and (28) of $\dim(2n_1\times 2n_2)$. Note that the accuracy of $\mathbf{P}(t)$ depends on accurate solutions of $\mathbf{L}$ and $\mathbf{H}$. The Kalman filter gain can then be obtained from (5).

The following algorithm presents a complete solution to our problem.

1. Use (30) with (29) to calculate $\mathbf{L}^{(i+1)}$, then solve (31) to get $\mathbf{H}^{(i)}$ recursively.
2. Calculate $\underline{\mathbf{U}}(t_0)$, $\underline{\mathbf{Z}}(t_0)$, $\underline{\mathbf{X}}(t_0)$ and $\underline{\mathbf{Y}}(t_0)$ from equations (33)–(36).
3. Calculate $\mathbf{U}(t)$, $\mathbf{Z}(t)$, $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ from equations (40)–(43).
4. Calculate $\mathbf{P}(t)$ from equation (44).

3. NUMERICAL EXAMPLE

In order to demonstrate the proposed method for the solution of the Kalman filter gains, we present results for a Kalman filter of the F-8 aircraft.[2,7]

The system matrices are

$$\mathbf{A}_1=\begin{pmatrix}0\cdot278386 & -0\cdot965256\\ 0\cdot089833 & -0\cdot290700\end{pmatrix},\qquad \mathbf{A}_2=\begin{pmatrix}-0\cdot074210 & 0\cdot016017\\ 0\cdot012815 & -0\cdot001398\end{pmatrix}$$

$$\mathbf{A}_3=\begin{pmatrix}-0\cdot001815 & 0\cdot005873\\ 0\cdot002850 & -0\cdot009223\end{pmatrix},\qquad \mathbf{A}_4=\begin{pmatrix}-0\cdot030344 & 0\cdot075024\\ -0\cdot075092 & -0\cdot016777\end{pmatrix}$$

$$\mathbf{C}_1=\begin{pmatrix}0 & 0\\ 1 & -3\cdot236\end{pmatrix},\qquad \mathbf{C}_2=\begin{pmatrix}0 & 0\cdot005\\ -0\cdot003152 & 0\cdot01302\end{pmatrix}$$

$$\mathbf{G}_1=\begin{pmatrix}-46\cdot626960\\ 7\cdot858776\end{pmatrix},\qquad \mathbf{G}_2=\begin{pmatrix}-18\cdot210002\\ -45\cdot049998\end{pmatrix}$$

The small parameter ε, the weighting matrix and the noise intensity matrix are $\varepsilon=0\cdot025$, $\mathbf{W}=0\cdot000315$ and $\mathbf{V}=\mathrm{diag}\{0\cdot000686, 40\}$ and the initial condition $\mathbf{P}(t_0)=\mathrm{diag}[0\cdot1, 0\cdot1, 1\cdot0, 1\cdot0]$. With the proposed method the simulation result for the singularly perturbed matrix differential equation (6) is obtained by using the package PC-MATLAB for the computer-aided control system design.[14]

After six iterations $\mathbf{L}^{(6)}$ and $\mathbf{H}^{(6)}$ are

$$\mathbf{L}^{(6)}=\begin{pmatrix}68\cdot2313 & -214\cdot6757 & -1\cdot2781 & -0\cdot1506\\ -10\cdot8173 & 33\cdot5637 & 0\cdot1372 & 0\cdot0239\\ -28\cdot6171 & 92\cdot2040 & 0\cdot611 & -0\cdot0029\\ 19\cdot3762 & -61\cdot8344 & 0\cdot6401 & -0\cdot1905\end{pmatrix}$$

$$\mathbf{H}^{(6)}=\begin{pmatrix}0\cdot5627 & 0\cdot8499 & 1\cdot093 & -0\cdot1018\\ 0\cdot0027 & -0\cdot0856 & 0\cdot1191 & -0\cdot0146\\ 25\cdot5713 & -16\cdot1464 & 60\cdot7209 & -9\cdot6570\\ -18\cdot0845 & 58\cdot3243 & -189\cdot9128 & 29\cdot9789\end{pmatrix}$$

The initial time is selected as $t_0 = 0$. When $t = 0\cdot 5$, using (44), we get

$$\mathbf{P}_{\rm app}(0\cdot 5) = \begin{pmatrix} 0\cdot 1699 & 0\cdot 0282 & 2\cdot 7041 & 2\cdot 5044 \\ 0\cdot 0282 & 0\cdot 0169 & 0\cdot 5829 & -0\cdot 0629 \\ 2\cdot 7041 & 0\cdot 5829 & 137\cdot 7923 & 73\cdot 6011 \\ 2\cdot 5044 & -0\cdot 0629 & 73\cdot 6011 & 85\cdot 6784 \end{pmatrix} \quad (45)$$

The obtained solution $\mathbf{P}_{\rm app}$ given by (45) is identical to the solution of the global Riccati differential equation (6) obtained by using any standard method.[12] However, in our method we have been using the reduced-order algorithm and the problem of ill-conditioning due to the singularly perturbed structure is eliminated. After getting the solution of (6), the Kalman filter gain can be obtained by (5) as

$$\mathbf{K}(0\cdot 5) = \mathbf{P}_{\rm app}(0\cdot 5)\mathbf{C}^{\rm T}\mathbf{V}^{-1} = \begin{pmatrix} 18\cdot 2536 & 0\cdot 0026 \\ -0\cdot 4586 & -0\cdot 0007 \\ 536\cdot 4514 & 0\cdot 0335 \\ 624\cdot 4778 & 0\cdot 0898 \end{pmatrix}$$

4. CONCLUSIONS

The Kalman filter gain of a singularly perturbed system is solved with any desired accuracy in terms of reduced-order equations. The proposed method considerably reduces the amount of computation required and is very suitable for parallel computations.

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