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Optimal output feedback control of discrete linear, singularly perturbed, stochastic systems

MUHAMMAD T. QURESHI†, XUEMIN SHEN† and ZORAN GAJIC†

The static output feedback control problem for discrete linear, singularly perturbed, stochastic systems is studied. A recursive algorithm is presented to solve the corresponding coupled non-linear algebraic equations. The algorithm removes the ill-conditioning by decomposing the higher order equations into lower order equations and develops a technique which enables us to obtain an arbitrary accuracy, that is, $O(\epsilon^j)$ ($j = 1, 2, 3, \ldots$) approximation for this problem. A numerical example is presented to support the theoretical results.

1. Introduction

The output feedback control problem, when a very limited number of state measurements are available for control implementation, received increasing attention in 1970s (e.g. Levine and Athans 1970, Mendel 1974, Ermer and Vandelinde 1973, Kurtaran and Sidar 1974, Kurtaran 1975). It also offers an important advantage in simplicity over controllers using full-state feedback. In most applications, the output measurements are contaminated with noise, mainly due to the measurement errors. Usually it is assumed that such a noise is white.

For continuous systems, the optimal solution for the noise-free output feedback problem is presented in several places in the literature (e.g. Levine and Athans 1970, Mendel 1974, Kurtaran and Sidar 1974, Moerder and Calise 1985). However, for noisy output, a major difficulty is encountered in finding the optimal solution if the classical quadratic performance index is used. Due to the presence of white noise, the performance index necessarily diverges (Ermer and Vandelinde 1973, Kurtaran and Sidar 1974), which necessitates the use of some alternative performance measure. It was shown by Ermer and Vandelinde (1973) that the discrete linear, stochastic output feedback control problem is well posed. Optimal solutions for discrete stochastic output feedback control problems presented in Ermer and Vandelinde (1973), Kurtaran (1975) are obtained in terms of high order non-linear algebraic equations.

Singularly perturbed output feedback systems did not receive much attention until 1980 (Calise and Moerder 1985, Chemouil and Wahdam 1980, Fossard and Magni 1980, Moerder and Calise 1985, 1988, Khalil 1981, 1987, Gajic *et al.* 1989), due to their inherent ill-conditioned dynamics. For noise-free, output feedback, continuous singularly perturbed systems, a well-defined recursive algorithm is developed in Gajic *et al.* (1989). The algorithm removes the inherent ill-conditioning

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for singularly perturbed systems by decomposing high order non-linear equations into low order algebraic equations corresponding to slow and fast modes.

In this paper a new recursive algorithm is developed for the discrete singularly perturbed output feedback stochastic control problem. Non-linear algebraic matrix equations are decomposed to ones corresponding to slow and fast modes, so that only low-order systems are involved in algebraic computations. Moreover, such a decomposition removes the ill-conditioning of the higher order system. The proposed algorithm gives the accuracy of $O(\varepsilon^j)$, where ε is a small perturbation parameter and j is the number of iterations.

The paper is organized as follows. Section 2 formulates the problem of optimal output feedback in terms of higher order algebraic equations. Section 3 describes the derivation for decomposing the higher order problem to lower order problems, and the recursive algorithm for solving these lower order equations. Section 4 shows the application of the algorithm through a real world example, and § 5 concludes the paper.

2. Problem formulation

A discrete linear singularly perturbed stochastic system is given by Gajic *et al.* (1990), Gajic and Shen (1991).

$$x_1(k+1) = (I + \varepsilon A_1)x_1(k) + \varepsilon A_2(k)x_2(k) + \varepsilon B_1 u(k) + \varepsilon G_1 w(k) \quad (1)$$

$$x_2(k+1) = A_3 x_1(k) + A_4 x_2(k) + B_2 u(k) + G_2 w(k) \quad (2)$$

$$y(k) = C_1 x_1(k) + C_2 x_2(k) + v(k) \quad (3)$$

where $x_1(k) \in R^{n_1}$, $x_2 \in R^{n_2}$ are state vectors, $u \in R^m$ is a control input, $y \in R^r$ is the measured output, $w \in R^s$ and $v \in R^r$ are stationary uncorrelated gaussian zero mean white noise processes with intensities $W > 0$ and $V > 0$, respectively. Matrices A_i, B_j, G_j, and C_j, $i = 1, 2, 3, 4$, $j = 1, 2$ are constant matrices of compatible dimensions.

With (1)–(3), consider the performance criterion

$$J = E\left(\sum_{k=0}^{\infty} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}^{\mathrm{T}} Q \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u^{\mathrm{T}}(k) R u(k)\right) \quad (4)$$

with positive definite matrix R and semipositive definite matrix Q, which has to be minimized. In addition, the control input $u(k)$ is constrained to

$$u(k) = Fy(k) \quad (5)$$

Equations (1)–(3) can be written in compact form as the following

$$x(k+1) = Ax(k) + Bu(k) + Gw(k) \quad (6)$$

$$y(k) = Cx(k) + v(k) \quad (7)$$

where

$$x(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}, \quad A = \begin{bmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon B_1 \\ B_2 \end{bmatrix}$$

$$G = \begin{bmatrix} \varepsilon G_1 \\ G_2 \end{bmatrix}, \quad C = [C_1 \quad C_2]$$

Substituting (7) into (5), we obtain

$$u(k) = FCx(k) + Fv(k) \tag{8}$$

The performance criterion, then becomes

$$J = E\left[\sum_{k=0}^{\infty} x^{\mathrm{T}}(k)Qx(k) + (FCx(k) + Fv(k))^{\mathrm{T}}R(FCx(k) + Fv(k))\right]$$

Interchanging the expectation and summation and using $E[v(k)x^{\mathrm{T}}(k)] = 0$, $E[v^{\mathrm{T}}(k)x(k)] = 0$, J can be written as

$$J = \sum_{k=0}^{\infty} (\mathrm{tr}\,[QP^*(k)] + \mathrm{tr}\,[C^{\mathrm{T}}F^{\mathrm{T}}RFCP^*(k)] + \mathrm{tr}\,[F^{\mathrm{T}}RFV(k)])$$

where $P^*(k) = E[x(k)x(k)^{\mathrm{T}}]$ is the variance of the system driven by white noise and can be obtained from (6).

For stochastic systems, the performance criterion, as described above, grows without bounds, but in the limit, the average value of J given by Halyo and Broussard (1981)

$$J = \lim_{N\to\infty} \frac{1}{2N}\left(\sum_{k=1}^{N} \mathrm{tr}\,[QP^*(k) + C^{\mathrm{T}}F^{\mathrm{T}}RFCP^*(k)] + \mathrm{tr}\,[F^{\mathrm{T}}RFV(k)]\right)$$

is bounded, provided that $P^*(k)$ is bounded and convergent. Such an average cost is given by

$$J = \mathrm{tr}\,[QP] + \mathrm{tr}\,[C^{\mathrm{T}}F^{\mathrm{T}}RFCP] + \mathrm{tr}\,[F^{\mathrm{T}}RFV] \tag{9}$$

where P is the steady state solution of $P^*(k)$ which can be found from (6) and (8) as follows

$$P = (A + BFC)P(A + BFC)^{\mathrm{T}} + BFVF^{\mathrm{T}}B^{\mathrm{T}} + GWG^{\mathrm{T}} \tag{10}$$

J can now be minimized with respect to F subject to constraint (10). Using the Lagrange multiplier technique the following equations are obtained in Ermer and Vandelinde (1973), Kurtaran (1975)

$$L = (A + BFC)^{\mathrm{T}}L(A + BFC) + C^{\mathrm{T}}F^{\mathrm{T}}RFC + Q \tag{11}$$

$$F = -(R^{\mathrm{T}} + B^{\mathrm{T}}LB)^{-1}B^{\mathrm{T}}LAPC^{\mathrm{T}}(CPC^{\mathrm{T}} + V^{\mathrm{T}})^{-1} \tag{12}$$

where L is the matrix of Lagrange multipliers. Equations (10)–(12) are high-order non-linear algebraic equations which have to be solved for P, L, and F. It is worth mentioning that the derivations for (10)–(12) assume that all the eigenvalues of the matrix $(A + BFC)$ have moduli strictly less than 1.

It is shown in Halyo and Broussard (1981) that the following algorithm proposed for the numerical solution of (10)–(12) converges to a local minimum under non-restrictive assumptions. The algorithm initially chooses $F^{(0)}$ such that $A + BF^{(0)}C$ is a stable matrix and then computes the following equations iteratively for $i = 1, 2, \ldots$

$$P^{(i+1)} = (A + BF^{(i)}C)P^{(i+1)}(A + BF^{(i)}C)^{\mathrm{T}} + BF^{(i)}VF^{(i)\mathrm{T}}B^{\mathrm{T}} + GWG^{\mathrm{T}} \tag{13}$$

$$L^{(i+1)} = (A + BF^{(i)}C)^{\mathrm{T}}L^{(i+1)}(A + BF^{(i)}C) + C^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C + Q \tag{14}$$

$$F_{\mathrm{new}} = -(R^{\mathrm{T}} + B^{\mathrm{T}}L^{(i+1)}B)^{-1}B^{\mathrm{T}}L^{(i+1)}AP^{(i+1)}C^{\mathrm{T}}(CP^{(i+1)}C^{\mathrm{T}} + V^{\mathrm{T}})^{-1} \tag{15}$$

$$F^{(i+1)} = F^{(i)} + \alpha_i(F_{\mathrm{new}} - F^{(i)}) \tag{16}$$

where $\alpha_i \in (0, 1]$ is chosen at each iteration to ensure that the minimum is not overshot, that is

$$J_{i+1} = \mathrm{tr}\,[(Q + C^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C)P^{(i+1)}] + \mathrm{tr}\,[F^{(i)\mathrm{T}}RF^{(i)}V]$$
$$< J_i = \mathrm{tr}\,[(Q + C^{\mathrm{T}}F^{(i-1)\mathrm{T}}RF^{(i-1)}C)P^{(i)}] + \mathrm{tr}\,[F^{(i-1)\mathrm{T}}RF^{(i-1)}V]$$

The next section shows how we can decompose (13)–(15) in the algebraic equations corresponding to slow and fast modes, in order to get lower order, well defined, algebraic equations, and to achieve any order of accuracy.

3. Lower order decomposition

Equation (14) is a standard Lyapunov equation of the discrete singularly perturbed linear system, while (13) is not in the standard form. Therefore, a slight difference occurs in the lower order expressions for these two equations. First, we will decompose (13).

Partition matrices $(A + BF^{(i)}C)$, $(BF^{(i)}VF^{(i)\mathrm{T}}B^{\mathrm{T}} + GWG^{\mathrm{T}})$ and $P^{(i)}$ as follows

$$A + BF^{(i)}C = \begin{bmatrix} I + \varepsilon D_1^{(i)} & \varepsilon D_2^{(i)} \\ D_3^{(i)} & D_4^{(i)} \end{bmatrix} \tag{17}$$

$$BF^{(i)}VF^{(i)\mathrm{T}}B^{\mathrm{T}} + GWG^{\mathrm{T}} = \begin{bmatrix} \varepsilon^2 S_1^{(i)} & \varepsilon S_2^{(i)} \\ \varepsilon S_2^{(i)\mathrm{T}} & S_3^{(i)} \end{bmatrix} \tag{18}$$

$$P^{(i)} = \begin{bmatrix} \varepsilon P_1^{(i)} & \varepsilon P_2^{(i)} \\ \varepsilon P_2^{(i)\mathrm{T}} & P_3^{(i)} \end{bmatrix}$$

where

$$D_1^{(i)} = A_1 + B_1F^{(i)}C_1$$
$$D_2^{(i)} = A_2 + B_1F^{(i)}C_2$$
$$D_3^{(i)} = A_3 + B_2F^{(i)}C_1$$
$$D_4^{(i)} = A_4 + B_2F^{(i)}C_2$$
$$S_1^{(i)} = B_1F^{(i)}VF^{(i)\mathrm{T}}B_1^{\mathrm{T}} + G_1WG_1^{\mathrm{T}}$$
$$S_2^{(i)} = B_1F^{(i)}VF^{(i)\mathrm{T}}B_2^{\mathrm{T}} + G_1WG_2^{\mathrm{T}}$$
$$S_3^{(i)} = B_2F^{(i)}VF^{(i)\mathrm{T}}B_2^{\mathrm{T}} + G_1WG_2^{\mathrm{T}}$$

With such partitions, expanding (13), we obtain

$$D_1^{(i)}P_1^{(i+1)} + P_1^{(i+1)}D_1^{(i)\mathrm{T}} + D_2^{(i)}P_2^{(i+1)\mathrm{T}} + P_2^{(i+1)}D_2^{(i)\mathrm{T}} + D_2^{(i)}P_3^{(i+1)}D_2^{(i)\mathrm{T}}$$
$$+ S_1^{(i)} + \varepsilon(D_1^{(i)}P_1^{(i+1)}D_1^{(i)\mathrm{T}} + D_2^{(i)}P_2^{(i+1)\mathrm{T}}D_1^{(i)\mathrm{T}} + D_1^{(i)}P_2^{(i+1)}D_2^{(i)\mathrm{T}}) = 0 \tag{19}$$

$$P_1^{(i+1)}D_3^{(i)\mathrm{T}} + P_2^{(i+1)}D_4^{(i)\mathrm{T}} + D_2^{(i)}P_3^{(i+1)}D_4^{(i)\mathrm{T}} - P_2^{(i+1)} + S_2^{(i)}$$
$$+ \varepsilon(D_1^{(i)}P_1^{(i+1)}D_3^{(i)\mathrm{T}} + D_2^{(i)}P_3^{(i+1)\mathrm{T}}D_3^{(i)\mathrm{T}} + D_1^{(i)}P_2^{(i+1)}D_4^{(i)\mathrm{T}}) = 0 \tag{20}$$

$$D_4^{(i)}P_3^{(i+1)}D_4^{(i)\mathrm{T}} - P_3^{(i)} + S_3^{(i)}$$
$$+ \varepsilon(D_3^{(i)}P_1^{(i+1)}D_3^{(i)\mathrm{T}} + D_3^{(i)}P_2^{(i+1)}D_4^{(i)\mathrm{T}} + D_4^{(i)}P_2^{(i+1)\mathrm{T}}D_3^{(i)\mathrm{T}}) = 0 \tag{21}$$

We can obtain an $O(\varepsilon)$ approximation of (19)–(21) by setting $\varepsilon = 0$ as

$$D_1^{(i)}\underline{P_1}^{(i+1)} + \underline{P_1}^{(i+1)}D_1^{(i)T} + D_2^{(i)}\underline{P_2}^{T(i+1)} + \underline{P_2}^{(i+1)}D_2^{(i)T} + S_1^{(i)} + D_2^{(i)}\underline{P_3}^{(i+1)}D_2^{(i)T} = 0 \tag{22}$$

$$\underline{P_1}^{(i+1)}D_3^{(i)T} + \underline{P_2}^{(i+1)}D_4^{(i)T} + D_2^{(i)}\underline{P_3}^{(i+1)}D_4^{(i)T} - \underline{P_2}^{(i+1)} + S_2^{(i)} = 0 \tag{23}$$

$$D_4^{(i)}\underline{P_3}^{(i+1)}D_4^{(i)T} - \underline{P_3}^{(i+1)} + S_3^{(i)} = 0 \tag{24}$$

where the underscore represents solutions when $\varepsilon = 0$.

From (23) we can express $\underline{P_2}^{(i+1)}$ in terms of $\underline{P_1}^{(i+1)}$ and $\underline{P_3}^{(i+1)}$ as

$$\begin{aligned}\underline{P_2}^{(i+1)} &= (\underline{P_1}^{(i+1)}D_3^{(i)T} + D_2^{(i)}\underline{P_3}^{(i+1)}D_4^{(i)T} + S_2^{(i)})(I - D_4^{(i)T})^{-1} \\ &= N_1^{(i)} + \underline{P_1}^{(i+1)}N_2^{(i)}\end{aligned} \tag{25}$$

where

$$N_1^{(i)} = (D_2^{(i)}\underline{P_3}^{(i+1)}D_4^{(i)T} + S_2^{(i)})(I - D_4^{(i)T})^{-1}$$

$$N_2^{(i)} = D_3^{(i)T}(I - D_4^{(i)T})^{-1}$$

Substituting (23) into (22) and doing some algebra, we obtain

$$\hat{A}^{(i)}\underline{P_1}^{(i+1)} + \underline{P_1}^{(i+1)}\hat{A}^{(i)T} + \hat{Q}^{(i)} = 0 \tag{26}$$

where

$$\hat{A}^{(i)} = D_1^{(i)} + D_2^{(i)}N_2^{(i)T}$$

$$\hat{Q}^{(i)} = D_2^{(i)}N_1^{(i)T} + N_1^{(i)}D_2^{(i)T} + D_2^{(i)}P_3^{(i+1)}D_2^{(i)T} + S_1^{(i)}$$

Since slow and fast subsystem feedback matrices $\hat{A}^{(i)}$ and $D_4^{(i)}$ are stable, equations (24), (26), and (25) can be solved sequentially for $\underline{P_3}^{(i+1)}$, $\underline{P_1}^{(i+1)}$, and $\underline{P_2}^{(i+1)}$ respectively.

In order to make an improvement over the $O(\varepsilon)$ approximate solutions given above, express $P_1^{(i+1)}$, $P_2^{(i+1)}$, and $P_3^{(i+1)}$ as

$$P_1^{(i+1)} = \underline{P_1}^{(i+1)} + \varepsilon E_1 \tag{27}$$

$$P_2^{(i+1)} = \underline{P_2}^{(i+1)} + \varepsilon E_2 \tag{28}$$

$$P_3^{(i+1)} = \underline{P_3}^{(i+1)} + \varepsilon E_3 \tag{29}$$

Subtract (22), (23), and (24) from (19), (20), and (21) respectively, and after doing some algebra, we obtain the following equations

$$D_4^{(i)}E_3D_4^{(i)T} - E_3 = -(D_3^{(i)}P_1^{(i+1)}D_3^{(i)T} + D_3^{(i)}P_2^{(i+1)}D_4^{(i)T} + D_4^{(i)}P_2^{(i+1)T}D_3^{(i)T}) \tag{30}$$

$$\begin{aligned}E_1\hat{A}^{(i)T} + \hat{A}^{(i)}E_1 = &-(HD_2^{(i)T} + D_2^{(i)}H^T + D_2^{(i)}E_3D_2^{(i)T} + D_1^{(i)}P_1^{(i+1)}D_1^{(i)T} \\ &+ D_2^{(i)}P_2^{(i+1)T}D_1^{(i)T} + D_1^{(i)}P_2^{(i+1)}D_2^{(i)T})\end{aligned} \tag{31}$$

$$E_2 = E_1D_3^{(i)T}(I - D_4^{(i)T})^{-1} + H \tag{32}$$

where

$$H = (D_2^{(i)}E_3D_4^{(i)T} + D_1^{(i)}P_1^{(i+1)}D_3^{(i)T} + D_2^{(i)}P_2^{(i+1)T}D_3^{(i)T} + D_1^{(i)}P_2^{(i+1)}D_4^{(i)T})(I - D_4^{(i)T})^{-1}$$

Equations (30), (31), and (32) can be solved sequentially for E_3, E_1, and E_2 by proposing the following algorithm.

Initialize $E_1^{(0)} = 0,\ E_2^{(0)} = 0,\ E_3^{(0)} = 0$

For $j = 0, 1, 2, \ldots$ do the following iterations

$$\begin{aligned} D_4^{(i)}E_3^{(j+1)}D_4^{(i)\mathrm{T}} - E_3^{(j+1)} = & -[D_3^{(i)}(\underline{P_1}^{(i+1)} + \varepsilon E_1^{(j)})D_3^{(i)\mathrm{T}} + D_3^{(i)}(\underline{P_2}^{(i+1)} \\ & + \varepsilon E_2^{(j)})D_4^{(i)\mathrm{T}} + D_4^{(i)}(\underline{P_2}^{(i+1)} + \varepsilon E_2^{(j)})^{\mathrm{T}} D_3^{(i)\mathrm{T}}] \end{aligned} \tag{33}$$

$$\begin{aligned} H^{(j)} = & [D_2^{(i)}E_3^{(j+1)}D_4^{(i)\mathrm{T}} + D_1^{(i)}(\underline{P_1}^{(i+1)} + \varepsilon E_1^{(j)})D_3^{(i)\mathrm{T}} \\ & + D_2^{(i)}(\underline{P_2}^{(i+1)} + \varepsilon E_2^{(j)})^{\mathrm{T}} D_3^{(i)\mathrm{T}} \\ & + D_1^{(i)}(\underline{P_2}^{(i+1)} + \varepsilon E_2^{(j)})D_4^{(i)\mathrm{T}}](I - D_4^{(i)\mathrm{T}})^{-1} \end{aligned}$$

$$\begin{aligned} E_1^{(j+1)}\hat{A}^{(i)\mathrm{T}} + \hat{A}^{(i)}E_1^{(j+1)} = & -[H^{(j)}D_2^{(i)\mathrm{T}} + D_2^{(i)}H^{(j)\mathrm{T}} + D_2^{(i)}E_3^{(j+1)}D_2^{(i)\mathrm{T}} + D_1^{(i)} \\ & + (\underline{P_1}^{(i+1)} + \varepsilon E_1^{(j)})D_1^{(i)\mathrm{T}} + D_2^{(i)}(\underline{P_2}^{(i+1)} + \varepsilon E_2^{(j)})^{\mathrm{T}} D_1^{(i)\mathrm{T}} \\ & + D_1^{(i)}(\underline{P_2}^{(i+1)} + \varepsilon E_2^{(j)})D_2^{(i)\mathrm{T}}] \end{aligned} \tag{34}$$

$$E_2^{(j+1)} = E_1^{(j+1)}D_3^{(i)\mathrm{T}}(I - D_4^{(i)\mathrm{T}})^{-1} + H^{(j)} \tag{35}$$

The following theorem establishes the features of the proposed algorithm.

Theorem 1

The algorithm described by (33)–(35) converges to the required solutions E_1, E_2, and E_3 with the rate of convergence $O(\varepsilon)$ (the proof is given in the Appendix), that is

$$\|E_m - E_m^{(j)}\| = O(\varepsilon^j), \quad m = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3, \ldots \tag{36}$$

□

The above theorem implies that

$$\|P_m^{(i+1)} - (\underline{P_m}^{(i+1)} + \varepsilon E_m^{(j)})\| = O(\varepsilon^j), \quad m = 1, 2, 3 \quad \text{and} \quad j = 1, 2, 3, \ldots$$

Therefore $P^{(i+1)}$ can be iteratively solved with an arbitrary accuracy.

Similar lower order equations can also be obtained for (14). Performing the following partitioning

$$Q + C^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C = \begin{bmatrix} q_1^{(i)} & q_2^{(i)} \\ q_2^{(i)\mathrm{T}} & q_3^{(i)} \end{bmatrix} \tag{37}$$

and

$$L^{(i)} = \begin{bmatrix} \varepsilon^{-1}L_1^{(i)} & L_2^{(i)} \\ L_2^{(i)\mathrm{T}} & L_3^{(i)} \end{bmatrix}$$

where

$$q_1^{(i)} = C_1^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C_1 + Q_1$$

$$q_2^{(i)} = C_1^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C_2 + Q_2$$

$$q_3^{(i)} = C_2^{\mathrm{T}}F^{(i)\mathrm{T}}RF^{(i)}C_2 + Q_3$$

the zero-order approximations of $L_1^{(i+1)}$, $L_2^{(i+1)}$, and $L_3^{(i+1)}$ can be found by solving the following equations

$$D_4^{(i)\mathrm{T}}\underline{L_3}^{(i+1)}D_4^{(i)} - \underline{L_3}^{(i+1)} = -q_3^{(i)} \tag{38}$$

$$\underline{L_1}^{(i+1)}\hat{A}^{(i)} + \hat{A}^{(i)\mathrm{T}}\underline{L_1}^{(i+1)} = -\hat{K}^{(i)} \tag{39}$$

$$\underline{L_2}^{(i+1)} = (\underline{L_1}^{(i+1)}D_2^{(i)} + D_3^{(i)\mathrm{T}}\underline{L_3}^{(i+1)}D_4^{(i)} + q_2^{(i)})(I - D_4^{(i)})^{-1} \tag{40}$$

where

$$\hat{K}^{(i)} = M_1^{(i)}D_3^{(i)} + D_3^{(i)\mathrm{T}}M_1^{(i)\mathrm{T}} + D_3^{(i)\mathrm{T}}\underline{L_3}^{(i+1)}D_3^{(i)} + q_1^{(i)}$$

$$M_1^{(i)} = (D_3^{(i)\mathrm{T}}\underline{L_3}^{(i-1)}D_4^{(i)} + q_2^{(i)})(I - D_4^{(i)\mathrm{T}})^{-1}$$

Defining the errors as

$$L_1^{(i+1)} = \underline{L_1}^{(i+1)} + \varepsilon\hat{E}_1 \tag{41}$$

$$L_2^{(i+1)} = \underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2 \tag{42}$$

$$L_3^{(i+1)} = \underline{L_3}^{(i+1)} + \varepsilon\hat{E}_3 \tag{43}$$

we obtain the following equations

$$D_3^{(i)\mathrm{T}}\hat{E}_3D_4^{(i)} - \hat{E}_3 = -(D_2^{(i)\mathrm{T}}L_1^{(i+1)}D_2^{(i)} + D_2^{(i)\mathrm{T}}L_2^{(i+1)}D_4^{(i)} + D_4^{(i)\mathrm{T}}L_2^{(i+1)\mathrm{T}}D_2^{(i)}) \tag{44}$$

$$\hat{E}_1\hat{A}^{(i)} + \hat{A}^{(i)\mathrm{T}}\hat{E}_1 = -(\hat{H}D_3^{(i)} + D_3^{(i)\mathrm{T}}\hat{H}^{\mathrm{T}} + D_3^{(i)\mathrm{T}}\hat{E}_3D_3^{(i)} + D_1^{(i)\mathrm{T}}L_1^{(i+1)}D_1^{(i)} + D_3^{(i)\mathrm{T}}L_2^{(i+1)\mathrm{T}}D_1^{(i)} + D_1^{(i)\mathrm{T}}L_2^{(i+1)}D_3^{(i)}) \tag{45}$$

$$\hat{E}_2 = \hat{E}_1D_2^{(i)}(I - D_4^{(i)})^{-1} + \hat{H} \tag{46}$$

where

$$\hat{H} = (D_3^{(i)\mathrm{T}}\hat{E}_3D_4^{(i)} + D_1^{(i)\mathrm{T}}L_1^{(i+1)}D_2^{(i)} + D_3^{(i)\mathrm{T}}L_2^{(i+1)\mathrm{T}}D_2^{(i)} + D_1^{(i)\mathrm{T}}L_2^{(i+1)}D_4^{(i)})(I - D_4^{(i)})^{-1}$$

Equations (44), (45), and (46) can be solved for $\hat{E}_3$, $\hat{E}_1$ and $\hat{E}_2$, respectively, by proposing a similar kind of algorithm as for (30), (31) and (32), as follows.

Initialize $\hat{E}_1^{(j)} = 0$, $\hat{E}_2^{(j)} = 0$, $\hat{E}_3^{(j)} = 0$.

For $j = 0, 1, 2, \ldots$ do the following iterations

$$D_4^{(i)\mathrm{T}}\hat{E}_3^{(j+1)}D_4^{(i)} - \hat{E}_3^{(j+1)} = -(D_2^{(i)\mathrm{T}}(\underline{L_1}^{(i+1)} + \varepsilon\hat{E}_1^{(j)})D_2^{(i)} + D_2^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})D_4^{(i)} + D_4^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})^{\mathrm{T}}D_2^{(i)} \tag{47}$$

$$\hat{H}^{(j)} = D_3^{(i)\mathrm{T}}\hat{E}_3^{(j+1)}D_4^{(i)} + D_1^{(i)\mathrm{T}}(\underline{L_1}^{(i+1)} + \varepsilon\hat{E}_1^{(j)})D_2^{(i)} + D_3^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})^{\mathrm{T}}D_2^{(i)} + D_1^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})D_4^{(i)})(I - D_4^{(i)})^{-1}$$

$$\hat{E}_1^{(j+1)}\hat{A}^{(i)} + \hat{A}^{(i)\mathrm{T}}\hat{E}_1^{(j+1)} = -(\hat{H}^{(j)}D_3^{(i)} + D_3^{(i)\mathrm{T}}\hat{H}^{(j)\mathrm{T}} + D_3^{(i)\mathrm{T}}\hat{E}_3^{(j+1)}D_3^{(i)} + D_1^{(i)\mathrm{T}}(\underline{L_1}^{(i+1)} + \varepsilon\hat{E}_1^{(j)})D_1^{(i)} + D_3^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})^{\mathrm{T}}D_1^{(i)} + D_1^{(i)\mathrm{T}}(\underline{L_2}^{(i+1)} + \varepsilon\hat{E}_2^{(j)})D_3^{(i)}) \tag{48}$$

$$\hat{E}_2^{(j+1)} = \hat{E}_1^{(j+1)}D_2^{(i)}(I - D_4^{(i)})_{-1} + \hat{H}^{(j)} \tag{49}$$

In summary, P can be computed by using equations (24)–(29), and (33)–(35), and L can be computed by using equations (38)–(43), and (47)–(49). Furthermore, since the algorithms for P and L are independent from each other, the computation can be done in parallel. The following algorithm presents a complete solution to our problem.

3.1. *Summary of the algorithm*

Step 1. Initialize $F^{(0)}$ such that $(A + BF^{(0)}C)$ is a stable matrix.

Step 2. For $i = 0, 1, 2, \ldots$ repeat steps 3–10.

Step 3. Calculate $D_1^{(i)}$, $D_2^{(i)}$, $D_3^{(i)}$, $D_4^{(i)}$ and $S_1^{(i)}$, $S_2^{(i)}$, $S_3^{(i)}$, and $q_1^{(i)}$, $q_2^{(i)}$, $q_3^{(i)}$ from (17), (18), and (36) respectively.

Step 4. Calculate $\underline{P_3}^{(i+1)}$, $\underline{P_1}^{(i+1)}$, and $\underline{P_2}^{(i+1)}$ from (24), (26), and (25) respectively.

Step 5. Initialize $E_1^{(0)} = 0$, $E_2^{(0)} = 0$, $E_3^{(0)} = 0$. For $j = 0, 1, 2, \ldots$ solve equations (33)–(35) until the desired accuracy is obtained.

Step 6. Construct $P_1^{(i+1)}$, $P_2^{(i+1)}$, $P_3^{(i+1)}$ from equations (27)–(29).

Step 7. Calculate $\underline{L_3}^{(i+1)}$, $\underline{L_1}^{(i+1)}$, $\underline{L_2}^{(i+1)}$, from (38), (39), and (40) respectively.

Step 8. Initialize $\hat{E}_1^{(0)} = 0$, $\hat{E}_2^{(0)} = 0$, $\hat{E}_3^{(0)} = 0$. For $j = 1, 2, \ldots$, solve equations (47)–(49) until the desired accuracy is achieved.

Step 9. Construct $L_1^{(i+1)}$, $L_2^{(i+1)}$, $L_3^{(i+1)}$ from equations (41)–(43).

Step 10. Construct P and L and calculate F_{new} and $F^{(i+1)}$ from (15) and (16), respectively.

If $F^{(0)}$ is chosen such that $A + BF^{(0)}C$ is stable then this algorithm converges for sufficiently small α such that $0 < \alpha \leqslant 1$ (Halyo and Broussard 1981). The effect of different values of α on the convergence speed and the convergence pattern, is shown in the following example.

4. Numerical example

In order to demonstrate the efficiency of the proposed algorithm, which yields $O(\varepsilon^j)$ approximation, we run a real world physical example (a fifth order discrete

$\varepsilon = 0{\cdot}27$, $\alpha = 0{\cdot}3$ (i)	$J^{(i)}$	$J^{(i)} - J^{(i)}_{\text{opt}}$
1	2·86059	0·28163
2	2·68072	0·10176
3	2·61881	0·03985
4	2·59517	0·01621
5	2·58570	0·00674
6	2·58178	0·00282
7	2·58014	0·00181
8	2·57945	$4{\cdot}9 \times 10^{-4}$
9	2·57915	$1{\cdot}9 \times 10^{-4}$
10	2·57902	6×10^{-5}
11	2·57897	1×10^{-5}
12	2·57896	0

Table 1. The $O(\varepsilon^j)$ approximation.

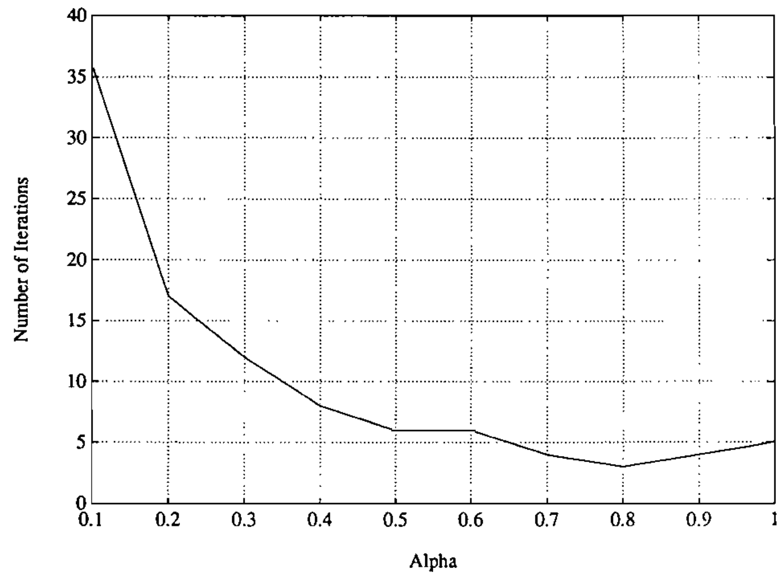

Figure 1. (*a*) Effect of α on the convergence speed.

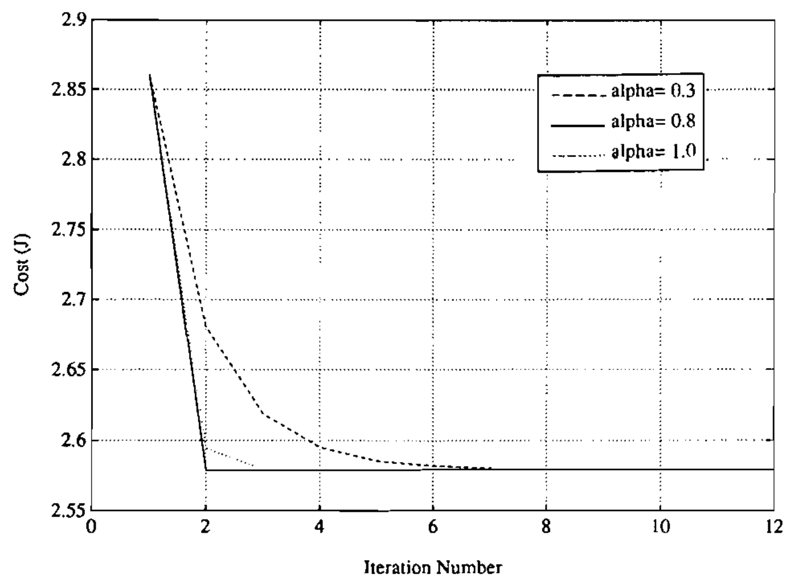

Figure 1. (*b*) Convergence behaviour for different α.

model of a steam power system (Mahmoud 1982). The problem matrices are as follows

$$A = \begin{bmatrix} 0.9150 & 0.0510 & 0.0380 & 0.0150 & 0.0380 \\ -0.0300 & 0.8890 & -0.0005 & 0.0460 & 0.1110 \\ -0.0060 & 0.4680 & 0.2470 & 0.0140 & 0.0480 \\ -0.7150 & -0.0220 & -0.0211 & 0.2400 & -0.0240 \\ -0.1480 & -0.0030 & -0.0040 & 0.0900 & 0.0260 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0098 \\ 0.1220 \\ 0.0360 \\ 0.5620 \\ 0.1150 \end{bmatrix}$$

The remaining matrices are chosen as

$$C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \quad Q = I_5, \quad R = 1, \quad G = B, \quad V = I_2, \quad W = 5.$$

The modulus of the eigenvalues of matrix A are 0·9, 0·9, 0·25, 0·25, 0·03. Thus we have two fast and three slow variables. The small parameter ε is chosen to be 0·27 which is roughly the ratio 0·25/0·9. The $O(\varepsilon^j)$ approximation is demonstrated in Table 1. The number of iterations for approximating P and L, as in steps (4) and (6) of the algorithm, are high enough so that $\|E_m^{(j+1)}\|_\infty - \|E_m^{(j)}\|_\infty$ and $\|\hat{E}_m^{(j+1)}\|_\infty - \|\hat{E}_m^{(j)}\|_\infty$, $m = 1, 2, 3$, are less than 10^{-5}, which verifies the $O(\varepsilon^j)$ theory. For this example the typical number of iterations for such tolerance was nine.

Figure 1(*a*) shows the effect of α on the convergence speed. It is noted that the convergence is fastest if α is chosen close to 0·85. Figure 1(*b*) shows the convergence behaviour for different α. Simulation results are obtained using a L-A-S package for computer aided control systems (West *et al.* 1985).

5. Conclusion

The reduced order numerical technique is obtained for the solution of the non-linear algebraic equations of the output feedback control problem of discrete singularly perturbed linear stochastic systems. It brings a considerable reduction in the size of computation and makes the problem well defined. In addition, the presented algorithm is very suitable for parallel programming, so that the processing time is also reduced.

Appendix

Proof of Theorem 1

Let $d_m^{(j)} = E_m^{(j)} - E_m^{(j-1)}$, $m = 1, 2, \ldots$, then (33) can be written as

$$D_4^{(i)} d_3^{(j+1)} D_3^{(i)\mathrm{T}} = -\varepsilon[D_3^{(i)} d_1^{(j)} D_3^{(i)\mathrm{T}} + D_3^{(i)} d_2^{(j)} D_4^{(i)\mathrm{T}} + D_4^{(i)} d_2^{(j)\mathrm{T}} D_3^{(i)\mathrm{T}}]$$

from which it follows that $\|d_3^{(j+1)}\| = O(\varepsilon)$.

Similarly

$$H^{(j)} - H^{(j-1)} = (D_2^{(i)} d_3^{(j+1)} D_4^{(i)\mathrm{T}} + D_1^{(i)} \varepsilon d_1^{(j)} D_3^{(i)\mathrm{T}} + D_2^{(i)} \varepsilon d_2^{(j)\mathrm{T}} D_3^{(i)\mathrm{T}} + D_1^{(i)} \varepsilon d_2^{(j)} D_4^{(i)\mathrm{T}})(I - D_4^{(i)\mathrm{T}})^{-1}$$

Since all terms on the right hand side are $O(\varepsilon)$, this implies that $\|H^{(j)} - H^{(j-1)}\| = O(\varepsilon)$. On the same lines (34) can be written as

$$d_1^{(j+1)}(\hat{A}^{(i)})^\mathrm{T} + \hat{A}^{(i)} d_1^{(j+1)} = -[(H^{(j)} - H^{(j-1)}) D_2^{(i)\mathrm{T}} + D_2^{(i)}(H^{(j)} - H^{(j-1)})^\mathrm{T} + D_2^{(i)} d_3^{(j+1)} D_2^{(i)\mathrm{T}} + D_1^{(i)} \varepsilon d_1^{(j)} D_1^{(i)\mathrm{T}} + D_2^{(i)} \varepsilon d_2^{(j)\mathrm{T}} D_1^{(i)\mathrm{T}} + D_1^{(i)} \varepsilon d_2^{(j)} D_2^{(i)\mathrm{T}}].$$

All terms on the right hand side are $O(\varepsilon)$, which implies that $\|d_1^{(j+1)}\| = O(\varepsilon)$. Similarly (35) can be written as

$$d_2^{(j+1)} = d_1^{(j+1)} D_3^{(i)\mathrm{T}}(I - D_4^{(i)\mathrm{T}})^{-1} + H^{(j)} - H^{(j-1)}$$

Again all terms on the right hand side are $O(\varepsilon)$, which implies that $\|d_2^{(j+1)}\| = O(\varepsilon)$.

Continuing the same procedure it follows that

$$\|E_m^{(j)} - E_m\| = O(\varepsilon^j), \quad m = 1, 2, 3; \quad j = 1, 2, \ldots$$

Thus the proposed algorithm is convergent.

Using $D_m^{(\infty)}$, $m = 1, 2, 3$ in (33)–(35) and comparing it to (30)–(32), implies that the algorithm (33)–(35) converges to the unique solutions of (30)–(32). This completes the proof. □

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