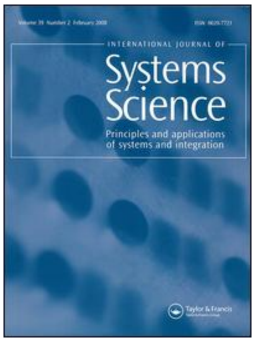

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Solution of the singularly perturbed matrix difference Riccati equation

XUEMIN SHEN†

A new method is introduced to obtain the solution of the singularly perturbed matrix difference Riccati equation by solving two reduced order linear equations. The order reduction is achieved via the use of the Chang's transformation applied to the hamiltonian matrix of a singularly perturbed linear-quadratic control problem. Since the decoupling transformation can be obtained up to an arbitrary degree of accuracy at very low cost, this approach produces an efficient numerical method for solving singularly perturbed difference Riccati equations. The results are verified through a real world example.

1. Introduction

A singularly perturbed linear discrete system is represented by (Litkouhi and Khalil 1985)

$$\left.\begin{aligned} x_1(k+1) &= (I + \varepsilon A_1)x_1(k) + \varepsilon A_2 x_2(k) + \varepsilon B_1 u(k) \\ x_2(k+1) &= A_3 x_1(k) + A_4 x_2(k) + B_2 u(k) \end{aligned}\right\} \tag{1}$$

with slow states $x_1 \in \mathbb{R}^{n_1}$, fast states $x_2 \in \mathbb{R}^{n_2}$ and control inputs $u \in \mathbb{R}^{n_i}$. The performance criterion of the corresponding linear-quadratic control problem is defined by

$$J_k = \tfrac{1}{2} x_N^{\mathrm{T}} F x_N + \tfrac{1}{2} \sum_{k=0}^{N-1} (x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k) \tag{2}$$

where

$$\left.\begin{aligned} Q &= \begin{bmatrix} Q_1 & Q_2 \\ Q_2^{\mathrm{T}} & Q_3 \end{bmatrix}, \quad Q \geqslant 0 \\ F &= \begin{bmatrix} F_1/\varepsilon & F_2 \\ F_2^{\mathrm{T}} & F_3 \end{bmatrix}, \quad R > 0 \end{aligned}\right\} \tag{3}$$

The coupled state and costate equations can be written as (Lewis 1986)

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^{\mathrm{T}} \\ QA^{-1} & A^{\mathrm{T}} + QA^{-1}BR^{-1}B^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} = H \begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} \tag{4}$$

where H is the symplectic matrix which has the property that the eigenvalues of H can be grouped into two disjoint subsets Γ_1 and Γ_2, such that for every $\lambda_c \in \Gamma_1$ there exists $\lambda_d \in \Gamma_2$, which satisfies $\lambda_c \times \lambda_d = 1$, and we can choose either Γ_1 or Γ_2 to contain only the stable eigenvalues (Salgado *et al.* 1988).

The optimal control law has a very well known form:

$$u_k = -R^{-1}B^{\mathrm{T}}\lambda_{k+1} = -R^{-1}B_k^{\mathrm{T}}P_{k+1}x_{k+1} \tag{5}$$

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† Rutgers University, Box 909, Department of Electrical and Computer Engineering, Piscataway, NJ 08855-0909, U.S.A.

where P_k satisfies the difference Riccati equation given by

$$\begin{aligned} P_k &= Q + A^{\mathrm{T}} P_{k+1}(I + SP_{k+1})^{-1} A \\ &= Q + A^{\mathrm{T}} P_{k+1} A - A^{\mathrm{T}} P_{k+1} B(R + B^{\mathrm{T}} P_{k+1} B)^{-1} B^{\mathrm{T}} P_{k+1} A \end{aligned} \tag{6}$$

where

$$A = \begin{bmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{bmatrix} \tag{7}$$

$$\left.\begin{aligned} S = BR^{-1}B^{\mathrm{T}} = \begin{bmatrix} \varepsilon^2 S_1 & \varepsilon Z \\ \varepsilon Z^{\mathrm{T}} & S_2 \end{bmatrix} \\ S_1 = B_1 R^{-1} B_1^{\mathrm{T}}, \quad S_2 = B_2 R^{-1} B_2^{\mathrm{T}}, \quad Z = B_1 R^{-1} B_2^{\mathrm{T}} \end{aligned}\right\} \tag{8}$$

The presence of a small parameter ε makes this problem numerically ill-defined (Likhouhi and Khalil 1985). In order to overcome this difficulty and obtain an efficient numerical method for solving (6), we will utilize the known hamiltonian form (4) of the solution of the difference Riccati equation and the non-singular Chang transformation (Chang 1972). The hamiltonian form can 'linearize' the difference Riccati equation and the Chang transformation is used to block diagonalize the hamiltonian, so that the required solution of the Riccati equation is obtained in terms of reduced order problems. An efficient Newton type algorithm (Gajic and Shen 1990, Gajic *et al.* 1990) (with the quadratic rate of convergence, that is, $O(\varepsilon^{2^i})$, where i is a number of iterations) is used for solving algebraic equations, which results in forming the Chang transformation.

2. Hamiltonian method for solving the singularly perturbed matrix difference Riccati equation

The solution of (6) can be sought in the form (see Appendix A)

$$P_k = M_k N_k^{-1} \tag{9}$$

where matrices M_k and N_k satisfy a system of linear equations

$$\left.\begin{aligned} N_k &= A^{-1} N_{k+1} + A^{-1} BR^{-1} B^{\mathrm{T}} M_{k+1} \\ M_k &= QA^{-1} N_{k+1} + (A^{\mathrm{T}} + QA^{-1} BR^{-1} B^{\mathrm{T}}) M_{k+1} \end{aligned}\right\} \tag{10}$$

with $M(N) = F$, $N(N) = I$.

The following lemma guarantees the existence of the invertible solution for N_k.

Lemma

If the triple $(A, B, \sqrt{Q})$ is stabilizable-observable then the matrix N_k, with $N(N) = I$ is invertible for any $k = 0, 1, 2, \ldots, N$.

The proof is obvious (Pappas *et al.* 1980) since the Riccati equation has a unique non-negative definite solution which means $P_k = M_k N_k^{-1} \geqslant 0$, where (M_k, N_k) is a set of eigenvectors of the stable eigenvalues of the hamiltonian matrix which implies that N_k is invertible.

The solution of (6) is properly scaled as (Litkouhi and Khalil 1984)

$$P_k = \begin{bmatrix} P_{1k/\varepsilon} & P_{2k} \\ P_{2k}^{\mathrm{T}} & P_{3k} \end{bmatrix}, \quad P(N) = F = \begin{bmatrix} F_1/\varepsilon & F_2 \\ F_2^{\mathrm{T}} & F_3 \end{bmatrix} \tag{11}$$

where $\dim P_1 = n_1 \times n_1$, $\dim P_3 = n_2 \times n_2$.

Let compatible partitions of matrices M_k and N_k be

$$M_k = \begin{bmatrix} M_{1k} & M_{2k} \\ M_{3k} & M_{4k} \end{bmatrix}, \quad N_k = \begin{bmatrix} N_{1k} & N_{2k} \\ N_{3k} & N_{4k} \end{bmatrix} \tag{12}$$

Partitioning (10), according to (12), will reveal a decoupled structure, that is, equations for M_1, M_3, N_1 and N_3 are independent of equations for M_2, M_4, N_2 and N_4 and vice versa.

$$\begin{bmatrix} N_{1k} \\ N_{3k} \\ M_{1k} \\ M_{3k} \end{bmatrix} = \begin{bmatrix} I + \varepsilon\overline{A_1} & \varepsilon\overline{A_2} & \varepsilon^2\overline{S_1} & \varepsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \varepsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I + \varepsilon\overline{A_{11}^{\mathrm{T}}} & \overline{A_{21}^{\mathrm{T}}} \\ \overline{Q_3} & \overline{Q_4} & \varepsilon\overline{A_{12}^{\mathrm{T}}} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} N_{1k+1} \\ N_{3k+1} \\ M_{1k+1} \\ M_{3k+1} \end{bmatrix} = H \begin{bmatrix} N_{1k+1} \\ N_{3k+1} \\ M_{1k+1} \\ M_{3k+1} \end{bmatrix} \tag{13}$$

$$\begin{bmatrix} N_{2k} \\ N_{4k} \\ M_{2k} \\ M_{4k} \end{bmatrix} = \begin{bmatrix} I + \varepsilon\overline{A_1} & \varepsilon\overline{A_2} & \varepsilon^2\overline{S_1} & \varepsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \varepsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I + \varepsilon\overline{A_{11}^{\mathrm{T}}} & \overline{A_{21}^{\mathrm{T}}} \\ \overline{Q_3} & \overline{Q_4} & \varepsilon\overline{A_{12}^{\mathrm{T}}} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} N_{2k+1} \\ N_{4k+1} \\ M_{2k+1} \\ M_{4k+1} \end{bmatrix} = H \begin{bmatrix} N_{2k+1} \\ N_{4k+1} \\ M_{2k+1} \\ M_{4k+1} \end{bmatrix} \tag{14}$$

(see Appendix B).

Interchanging the second and third rows in (13) and (14), respectively, produces

$$\begin{bmatrix} N_{1k} \\ \varepsilon M_{1k} \\ N_{3k} \\ M_{3k} \end{bmatrix} = \begin{bmatrix} I + \varepsilon\overline{A_1} & \varepsilon\overline{S_1} & \varepsilon\overline{A_2} & \varepsilon\overline{S_2} \\ \varepsilon\overline{Q_1} & I + \varepsilon\overline{A_{11}^{\mathrm{T}}} & \varepsilon\overline{Q_2} & \varepsilon\overline{A_{21}^{\mathrm{T}}} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^{\mathrm{T}}} & \overline{Q_4} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} N_{1k+1} \\ \varepsilon M_{1k+1} \\ N_{3k+1} \\ M_{3k+1} \end{bmatrix}$$

$$= \begin{bmatrix} I + \varepsilon T_1 & \varepsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} N_{1k+1} \\ \varepsilon M_{1k+1} \\ N_{3k+1} \\ M_{3k+1} \end{bmatrix} \tag{15}$$

$$\begin{bmatrix} N_{2k} \\ \varepsilon M_{2k} \\ N_{4k} \\ M_{4k} \end{bmatrix} = \begin{bmatrix} I + \varepsilon\overline{A_1} & \varepsilon\overline{S_1} & \varepsilon\overline{A_2} & \varepsilon\overline{S_2} \\ \varepsilon\overline{Q_1} & I + \varepsilon\overline{A_{11}^{\mathrm{T}}} & \varepsilon\overline{Q_2} & \varepsilon\overline{A_{21}^{\mathrm{T}}} \\ \overline{A_3} & \overline{S_3} & \overline{A_4} & \overline{S_4} \\ \overline{Q_3} & \overline{A_{12}^{\mathrm{T}}} & \overline{Q_4} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \begin{bmatrix} N_{2k+1} \\ \varepsilon M_{2k+1} \\ N_{4k+1} \\ M_{4k+1} \end{bmatrix}$$

$$= \begin{bmatrix} I + \varepsilon T_1 & \varepsilon T_2 \\ T_3 & T_4 \end{bmatrix} \begin{bmatrix} N_{2k+1} \\ \varepsilon M_{2k+1} \\ N_{4k+1} \\ M_{4k+1} \end{bmatrix} \tag{16}$$

where

$$T_1 = \begin{bmatrix} \overline{A_1} & \overline{S_1} \\ \overline{Q_1} & \overline{A_{11}^{\mathrm{T}}} \end{bmatrix}, \quad T_2 = \begin{bmatrix} \overline{A_2} & \overline{S_2} \\ \overline{Q_2} & \overline{A_{21}^{\mathrm{T}}} \end{bmatrix},$$
$$T_3 = \begin{bmatrix} \overline{A_3} & \overline{S_3} \\ \overline{Q_3} & \overline{A_{12}^{\mathrm{T}}} \end{bmatrix}, \quad T_4 = \begin{bmatrix} \overline{A_4} & \overline{S_4} \\ \overline{Q_4} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \tag{17}$$

Introducing a notation

$$U_k = \begin{bmatrix} N_{1k} \\ \varepsilon M_{1k} \end{bmatrix}, \quad V_k = \begin{bmatrix} N_{3k} \\ M_{3k} \end{bmatrix}, \quad X_k = \begin{bmatrix} N_{2k} \\ \varepsilon M_{2k} \end{bmatrix}, \quad Y_k = \begin{bmatrix} N_{4k} \\ M_{4k} \end{bmatrix} \tag{18}$$

we obtain two systems of singularly perturbed difference equations

$$\left.\begin{aligned} U_k &= (I + \varepsilon T_1)U_{k+1} + \varepsilon T_2 V_{k+1}, \\ V_k &= T_3 U_{k+1} + T_4 V_{k+1} \end{aligned}\right\} \tag{19}$$

$$\left.\begin{aligned} X_k &= (I + \varepsilon T_1)X_{k+1} + \varepsilon T_2 Y_{k+1}, \\ Y_k &= T_3 X_{k+1} + T_4 Y_{k+1} \end{aligned}\right\} \tag{20}$$

with terminal conditions

$$U(N) = \begin{bmatrix} I \\ F_1 \end{bmatrix}, \quad V(N) = \begin{bmatrix} 0 \\ F_2^{\mathrm{T}} \end{bmatrix}, \quad X(N) = \begin{bmatrix} 0 \\ \varepsilon F_2 \end{bmatrix}, \quad Y(N) = \begin{bmatrix} I \\ F_3 \end{bmatrix} \tag{21}$$

Note that systems (19) and (20) have exactly the same form and the only difference is the terminal conditions.

Applying Chang's transformation (Chang 1972)

$$J = \begin{bmatrix} I - \varepsilon ML & -\varepsilon M \\ L & I \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} I & \varepsilon M \\ -L & I - \varepsilon LM \end{bmatrix} \tag{22}$$

where L and M satisfy

$$\left.\begin{aligned} 0 &= M + T_2 - MT_4 + \varepsilon(T_1 - T_2 L)M - \varepsilon MLT_2 \\ 0 &= -L + T_4 L - T_3 - \varepsilon L(T_1 - T_2 L) \end{aligned}\right\} \tag{23}$$

to (19) and (20) produces

$$\left.\begin{aligned} \overline{U_k} &= (I + \varepsilon T_1 - \varepsilon T_2 L)\overline{U_{k+1}} \\ \overline{V_k} &= (T_4 + \varepsilon L T_2)\overline{V_{k+1}} \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \overline{X_k} &= (I + \varepsilon T_1 - \varepsilon T_2 L)\overline{X_{k+1}} \\ \overline{Y_k} &= (T_4 + \varepsilon L T_2)\overline{Y_{k+1}} \end{aligned}\right\} \tag{25}$$

where

$$\overline{U}(N) = (I - \varepsilon ML)U(N) - \varepsilon MV(N), \quad \overline{V}(N) = LU(N) + V(N)$$
$$\overline{X}(N) = (I - \varepsilon ML)X(N) - \varepsilon MY(N), \quad \overline{Y}(N) = LX(N) + Y(N) \tag{26}$$

Matrices L and M can be obtained by using the recursive algorithm from Gajic *et al.* (1990). Solutions of (24), (25) are given by

$$\left.\begin{aligned} \overline{U_k} &= (I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{U}(N) \\ \overline{V_k} &= (T_4 + \varepsilon L T_2)^{N-k}\overline{V}(N) \\ \overline{X_k} &= (I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{X}(N) \\ \overline{Y_k} &= (T_4 + \varepsilon L T_2)^{N-k}\overline{Y}(N) \end{aligned}\right\} \tag{27}$$

The solutions in the original coordinates are

$$\begin{aligned} U_k &= (I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{U}(N) + \varepsilon M (T_4 + \varepsilon L T_2)^{N-k}\overline{V}(N) \\ V_k &= -L(I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{U}(N) + (I - \varepsilon L M)(T_4 + \varepsilon L T_2)^{N-k}\overline{V}(N) \\ X_k &= (I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{X}(N) + \varepsilon M (T_4 + \varepsilon L T_2)^{N-k}\overline{Y}(N) \\ Y_k &= -L(I + \varepsilon T_1 - \varepsilon T_2 L)^{N-k}\overline{X}(N) + (I - \varepsilon L M)(T_4 + \varepsilon L T_2)^{N-k}\overline{Y}(N) \end{aligned} \tag{28}$$

Partitioning (28) according to (18) will produce all the components of matrices M_k and N_k, that is

$$\left.\begin{aligned} \begin{bmatrix} N_{1k} \\ \varepsilon M_{1k} \end{bmatrix} &= \begin{bmatrix} U_{1k} \\ U_{2k} \end{bmatrix} = U_k, \quad \begin{bmatrix} N_{3k} \\ M_{3k} \end{bmatrix} = \begin{bmatrix} V_{1k} \\ V_{2k} \end{bmatrix} = V_k \\ \begin{bmatrix} N_{2k} \\ \varepsilon M_{2k} \end{bmatrix} &= \begin{bmatrix} X_{1k} \\ X_{2k} \end{bmatrix} = X_k, \quad \begin{bmatrix} N_{4k} \\ M_{4k} \end{bmatrix} = \begin{bmatrix} Y_{1k} \\ Y_{2k} \end{bmatrix} = Y_k \end{aligned}\right\} \tag{29}$$

Then the required solution of (6) is given by

$$P_k = \begin{bmatrix} U_{2k}/\varepsilon & X_{2k}/\varepsilon \\ V_{2k} & Y_{2k} \end{bmatrix} \begin{bmatrix} U_{1k} & X_{1k} \\ V_{1k} & Y_{1k} \end{bmatrix}^{-1} \tag{30}$$

Thus, in order to obtain the solutions of (6), that is P_k, which has dim $n \times n = (n_1 + n_2) \times (n_1 + n_2)$, we only solve two simple algebraic equations (23) of dimensions $(2n_2 \times 2n_1)$ and $(2n_1 \times 2n_2)$, respectively. The existing numerical algorithms for solving (23) are given by Gajic *et al.* (1990) where the rate of convergence is $O(\varepsilon^{2^i})$ (i is a number of iterations). Finally, the inversion of the matrix N_k has to be performed.

3. Numerical example

In order to demonstrate the proposed method, a linearized model of an F-8 aircraft (Litkouhi 1983) in the singularly perturbed continuous form (fast time version) is studied, with the system matrix

$$\begin{bmatrix} -0{\cdot}015 & -0{\cdot}0805 & -0{\cdot}001\,1666 & 0 \\ 0 & 0 & 0 & 0{\cdot}033\,33 \\ -2{\cdot}28 & 0 & -0{\cdot}84 & 1 \\ 0{\cdot}6 & 0 & -4{\cdot}8 & -0{\cdot}49 \end{bmatrix} \tag{31}$$

and the control matrix

$$\begin{bmatrix} -0.0000916 & 0.0007416 \\ 0 & 0 \\ -0.11 & 0 \\ -8.7 & 0 \end{bmatrix} \tag{32}$$

The first two rows represent two slow variables and the last two rows represent fast variables. The small perturbation parameter ε is chosen as 1/30. This model is discretized by Litkouhi (1983) using the sampling period $T = 1$, leading to

$$A = \begin{bmatrix} 0.98475 & -0.079903 & 0.0009054 & -0.0010765 \\ 0.041588 & 0.99899 & -0.035855 & 0.01284 \\ -0.54662 & 0.044916 & -0.32991 & 0.19318 \\ 2.6624 & -0.10045 & -0.92455 & -0.26325 \end{bmatrix} \tag{33}$$

$$B = \begin{bmatrix} 0.0037112 & 01000736 1 \\ -0.087051 & 0.0000093411 \\ -1.19844 & -0.00041378 \\ -3.1927 & 0.00092535 \end{bmatrix} \tag{34}$$

The remaining matrices are chosen as $R = I_2$, $Q = 10^{-2} I_4$ and the terminal condition $P(N) = F = \mathrm{diag}[0.5, 0.5, 0.01, 0.01]$.

With the proposed method, simulation results for the L equation (23) and the singularly perturbed matrix difference Riccati equation (6) are obtained by using the package L-A-S for the computer aided control system design (West *et al.* 1985) where error is defined as $\|L^{(i+1)} - L^{(i)}\|_\infty$.

i	Error
0	1·004359E−1
1	2·0066535E−2
2	1·445678E−3
3	1·09863E−4
4	7·74528E−6
5	4·91643E−7

Approximate solution of the L equation.

$$P_{\mathrm{app}} = \begin{bmatrix} 0.88976 & -0.077469 & -0.015048 & 0.00014416 \\ -0.077469 & 0.55719 & -0.016686 & 0.0047727 \\ -0.015048 & -0.016686 & 0.019299 & -0.0029608 \\ 0.00014416 & 0.0047727 & -0.0029608 & 0.011119 \end{bmatrix} \tag{35}$$

The terminal time is selected as $N = 8$ and k equals 4. The solution P_{app}, given by (35), is identical to the solution of the global Riccati difference equation (6) obtained using any standard method (Pappas *et al.* 1980). However, in our method we have been using the reduced-order algorithm and the problem of ill-conditioning due to the singularly perturbed structure is eliminated.

4. Conclusions

The singularly perturbed matrix difference Riccati equation is solved with any desired accuracy in terms of the reduced order equations. The proposed method considerably reducese the amount of computation required and is very well suited for parallel computations.

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The author wishes to express his thanks to Professor Zoran Gajic, without whose help this paper would not have been written.

Appendix A

Rewrite (10) (Pappas *et al.* 1980)

$$\begin{bmatrix} I & BR^{-1}B^{\mathrm{T}} \\ 0 & A^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} N_{k+1} \\ M_{k+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -Q & I \end{bmatrix} \begin{bmatrix} N_k \\ M_k \end{bmatrix} \tag{A 1}$$

which implies the following two relations:

$$\left.\begin{aligned} AN_k &= N_{k+1} + BR^{-1}B^{\mathrm{T}}M_{k+1} \\ -QN_k + M_k &= A^{\mathrm{T}}M_{k+1} \end{aligned}\right\} \tag{A 2}$$

Then

$$\left.\begin{aligned} A &= N_{k+1}N_k^{-1} + BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} \\ A^{\mathrm{T}} &= N_k^{-\mathrm{T}}N_{k+1}^{\mathrm{T}} + N_k^{-\mathrm{T}}M_{k+1}^{\mathrm{T}}BR^{-1}B^{\mathrm{T}} \\ M_kN_k^{-1} &= A^{\mathrm{T}}M_{k+1}N_k^{-1} + Q \end{aligned}\right\} \tag{A 3}$$

Assuming that N_k is invertable, substitute (A 3) in (6). We obtain

$$\begin{aligned} & A^{\mathrm{T}}P_{k+1}A - A^{\mathrm{T}}P_{k+1}B(R + B^{\mathrm{T}}P_{k+1}B)^{-1}B^{\mathrm{T}}P_{k+1}A + Q \\ &= A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}(N_{k+1}N_k^{-1} + BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1}) + M_kN_k^{-1} - A^{\mathrm{T}}M_{k+1}N_k^{-1} \\ &\quad - A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B(R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1}B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}A \\ &= A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} - A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B \\ &\quad \times (R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1}B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}A + M_kN_k^{-1} \\ &= A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} - A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B \\ &\quad \times (R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1}B^{\mathrm{T}}M_{k+1}(N_k^{-1} + N_{k+1}^{-1}BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1}) \\ &\quad + M_kN_k^{-1} \\ &= A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} - A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B \\ &\quad \times (R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} - A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B \\ &\quad \times (R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1}B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}BR^{-1}B^{\mathrm{T}}M_{k+1}N_k^{-1} + M_kN_k^{-1} \\ &= A^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B(R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)^{-1} \\ &\quad \times [(R + B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}B)R^{-1} - I - B^{\mathrm{T}}M_{k+1}N_{k+1}^{-1}BR^{-1}] \\ &\quad \times B^{\mathrm{T}}M_{k+1}N_k^{-1} + M_kN_k^{-1} = M_kN_k^{-1} = P_k \end{aligned} \tag{A 4}$$

Appendix B

From (4)

$$H = \begin{bmatrix} A^{-1} & A^{-1}BR^{-1}B^{\mathrm{T}} \\ QA^{-1} & A^{\mathrm{T}} + QA^{-1}BR^{-1}B^{\mathrm{T}} \end{bmatrix} \tag{B 1}$$

Since A^{-1} has same structure as A, that is

$$A^{-1} = \begin{bmatrix} I + O(\varepsilon) & O(\varepsilon) \\ O(1) & O(1) \end{bmatrix} \tag{B 2}$$

then

$$QA^{-1} = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} I + O(\varepsilon) & O(\varepsilon) \\ O(1) & O(1) \end{bmatrix} = \begin{bmatrix} O(1) & O(1) \\ O(1) & O(1) \end{bmatrix}$$

$$A^{-1}BR^{-1}B^{\mathrm{T}} = \begin{bmatrix} I + O(\varepsilon) & O(\varepsilon) \\ O(1) & O(1) \end{bmatrix} \begin{bmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(1) \end{bmatrix} = \begin{bmatrix} O(\varepsilon^2) & O(\varepsilon) \\ O(\varepsilon) & O(1) \end{bmatrix}$$

$$A^{\mathrm{T}} + QA^{-1}BR^{-1}B^{\mathrm{T}} = \begin{bmatrix} I + O(\varepsilon) & O(1) \\ O(\varepsilon) & O(1) \end{bmatrix}$$

$$H = \begin{bmatrix} I + \varepsilon\overline{A_1} & \varepsilon\overline{A_2} & \varepsilon^2\overline{S_1} & \varepsilon\overline{S_2} \\ \overline{A_3} & \overline{A_4} & \varepsilon\overline{S_3} & \overline{S_4} \\ \overline{Q_1} & \overline{Q_2} & I + \varepsilon\overline{A_{11}^{\mathrm{T}}} & \overline{A_{21}^{\mathrm{T}}} \\ \overline{Q_3} & \overline{Q_4} & \varepsilon\overline{A_{12}^{\mathrm{T}}} & \overline{A_{22}^{\mathrm{T}}} \end{bmatrix} \tag{B 3}$$

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