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Brief Paper

Study of the Discrete Singularly Perturbed Linear-quadratic Control Problem by a Bilinear Transformation*

ZORAN GAJIC†‡ and XUE-MIN SHEN†

Key Words—Large scale system; system order reduction; singular perturbation; linear optimal regulator; discrete time systems.

Abstract—This paper presents a new approach in the study of the linear quadratic control problem of singularly perturbed discrete systems. By applying a bilinear transformation, the algebraic discrete Riccati equation is converted into a continuous one, which can be solved by using the reduced-order recursive method already documented in the control literature. This method produces the reduced-order near-optimal solution up to an arbitrary order of accuracy and reduces the size of required computations. The method is very suitable for parallel programming. A real world example, an F-8 aircraft, demonstrates the efficiency of the proposed method.

1. *Introduction*

THE LINEAR singularly perturbed discrete systems have been studied recently in different set-ups by many researchers. Two main structures of singularly perturbed linear discrete systems have been considered: the fast (Butuzov and Vasileva, 1971; Hoppensteadt and Miranker, 1977; Blankenship, 1981; Litkouhi, 1983; Litkouhi and Khalil, 1984, 1985; Mahmoud, 1986; Oloomi and Sawan, 1987) and slow time-scale versions (Phillips, 1980; Naidu and Rao, 1985). Discrete time models of the singularly perturbed linear systems, similar to Phillips (1980) and Naidu and Rao (1985), were studied by Mahmoud and his coworkers (Mahmoud *et al.*, 1986). Since the slow time-scale version presupposes the asymptotic stability of the fast modes, it seems that in the design procedure of stabilizing feedback controllers, the fast time-scale version is much more appropriate (Litkouhi and Khalil, 1985). In this paper, we will adopt the structure of singularly perturbed discrete linear systems defined by Litkouhi and Khalil, and study the corresponding linear-quadratic control problem. We will take a new approach, based on a bilinear transformation (Kondo and Furuta, 1986).

It is known that the main equation of the optimal linear control theory, the Riccati equation, has a quite complicated form in the discrete time domain. Partitioning this equation, in the spirit of singular perturbation methodology, will produce a lot of terms and make corresponding problems numerically inefficient, even though problem order reduction is achieved. By applying a bilinear transformation, the solution of the discrete algebraic Riccati equation of singularly perturbed systems is obtained by using already known results for the corresponding continuous-time algebraic Riccati equation. The proposed method produces the reduced-order near-optimal solution, up to an arbitrary degree of accuracy $O(\varepsilon^k)$,§ where ε is a small positive perturbation parameter. The method reduces the size of required computations and is very suitable for parallel programming. A real world example, an F-8 aircraft, demonstrates the efficiency of the method introduced.

The importance of the existence of the $O(\varepsilon^k)$ theory, for small parameter problems, is indicated in Gajic *et al.* (1989) and Shen and Gajic (1990). It was shown in Gajic *et al.* (1989) that the $O(\varepsilon)$ theory fails to produce required results. In Shen and Gajic (1990) the approximate filter has to be obtained with the accuracy of at least $O(\varepsilon^6)$ in order to stabilize the plant-filter augmented system.

2. *Reduced-order near-optimal solution of the discrete algebraic Riccati equation of singularly perturbed systems*

The positive semi-definite stabilizing solution of the algebraic Riccati equation produces the answer to the optimal linear-quadratic steady state control problem; namely, a quadratic criterion

$$J = \sum_{k=0}^{\infty} (x^{\mathrm{T}}(k)Qx(k) + u^{\mathrm{T}}(k)Ru(k)) \tag{1}$$

is minimized along trajectories of a linear dynamic discrete system

$$x(k+1) = Ax(k) + Bu(k) \tag{2}$$

by using the control input in the form

$$u(k) = -(R + B^{\mathrm{T}}PB)^{-1}B^{\mathrm{T}}PAx(k) \tag{3}$$

where P is obtained from the algebraic Riccati equation (Dorato and Levis, 1971)

$$\begin{gathered} P = A^{\mathrm{T}}PA + Q - A^{\mathrm{T}}PB(B^{\mathrm{T}}PB + R)^{-1}B^{\mathrm{T}}PA, \\ R > 0, \quad Q \geq 0. \end{gathered} \tag{4}$$

For the singularly perturbed discrete systems, corresponding matrices are partitioned as Litkouhi (1983) and Litkouhi and Khalil (1984, 1985):

$$A = \begin{pmatrix} I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & A_4 \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon B_1 \\ B_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_1 & Q_2 \\ Q_2^{\mathrm{T}} & Q_3 \end{pmatrix} \tag{5}$$

where ε is a small positive singular perturbation parameter. In addition, the following condition is satisfied (Litkouhi and

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† Rutgers University, Department of Electrical and Computer Engineering, Piscataway, NJ 08855-0909, U.S.A.

‡ Author to whom all correspondence should be addressed.

§ $O(\varepsilon^k)$ stands for $C\varepsilon^k$, where C is a bounded constant and k is any arbitrary constant.

Khalil, 1985)

$$\det(I - A_4) \neq 0. \quad (6)$$

Due to the special structure of the problem matrices the required solution P has the form (Likhouhi and Khalil, 1984)

$$P = \begin{pmatrix} P_1/\varepsilon & P_2 \\ P_2^T & P_3 \end{pmatrix}. \quad (7)$$

The main goal in the theory of singular perturbations is to obtain the required solution in terms of the reduced-order problems, namely subsystems. In the case of the algebraic singularly perturbed discrete Riccati equation (4), the expansion of the partitioned matrix products will produce a lot of terms and make corresponding approach computationally very involved even though one is faced with the reduced order numerical problems.

In order to overcome this problem, we have used a bilinear transformation introduced in Kondo and Furuta (1986) to transform the discrete algebraic Riccati equation (4) into the continuous-time algebraic Riccati equation of the form

$$A_c^{fT}P + PA_c^f + Q_c^f - PS_c^fP = 0, \quad S_c^f = B_c^f R_c^{f-1} B_c^T \quad (8)$$

such that the solution of (4) is equal to the solution of (8).

It can be verified that the equation (8) preserves the structure of singularly perturbed systems. This equation can be solved in terms of the reduced-order problems very efficiently by using the recursive method developed in Gajic (1986) and Gajic *et al.* (1990), which converges with the rate of convergence of $O(\varepsilon)$.

Since the proposed method combines features of the bilinear transformation (Kondo and Furuta, 1986) and the recursive algorithm (Gajic, 1986; Gajic *et al.*, 1990), we will briefly summarize their main results.

The bilinear transformation states that equations (4) and (8) have the same solution if the following relations hold (Kondo and Furuta, 1986)

$$A_c^f = I - 2D^{-T} \quad (9a)$$

$$S_c^f = 2(I+A)^{-1}S_dD^{-1}, \quad S_d = BR^{-1}B^T \quad (9b)$$

$$Q_c^f = 2D^{-1}Q(I+A)^{-1} \quad (9c)$$

$$D = (I + A^T) + Q(I+A)^{-1}S_d \quad (9d)$$

assuming that $(I+A)^{-1}$ exists. Note that a superscript f stands for the fast time scale representation. It can be easily seen that the matrix

$$I + A = \begin{pmatrix} 2I + \varepsilon A_1 & \varepsilon A_2 \\ A_3 & I + A_4 \end{pmatrix} \quad (10)$$

is invertible for small values of ε if and only if the matrix $I + A_4$ is invertible. This condition is satisfied under assumption given in (6). It is important to point out that the matrix D defined in (9d) is nonsingular if (6) holds (Bar-Ness and Halbersberg, 1980).

Since there is no difference in the use of either the slow or fast time-scale representation for the continuous-time LQ control problem of singularly perturbed systems, we will adopt the slow time-scale version for this problem [It is customary to represent continuous time singularly perturbed systems by their slow time version (Kokotovic and Khalil, 1986; Kokotovic *et al.*, 1986).]

The slow time version of (9) can be obtained by multiplying the matrix A_c^f by $1/\varepsilon$ and matrix S_c^f by $1/\varepsilon^2$. Introducing a notation for the compatible partition of these matrices we have

$$A_c = \begin{pmatrix} A_{11} & A_{12} \\ A_{21}/\varepsilon & A_{22}/\varepsilon \end{pmatrix}, \quad S_c = \begin{pmatrix} S_{11} & S_{12}/\varepsilon \\ S_{12}^T/\varepsilon & S_{22}/\varepsilon^2 \end{pmatrix}. \quad (11)$$

By doing this, the required solution P, obtained now from (8), will be multiplied by ε, that is

$$\varepsilon P = P_c = \begin{pmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & \varepsilon P_3 \end{pmatrix}. \quad (12)$$

Going from the fast time version to slow time version does not change the matrix Q_c. It is partitioned as

$$Q_c = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} = Q_c^f. \quad (13)$$

Note that partitions defined in (11)–(13) have to be performed by a computer only, in the process of calculations, and there is no need for the corresponding analytical expressions.

The solution of (8) can be found in terms of the reduced-order problems by imposing standard stabilizability-detectability assumptions on the slow and fast subsystems. An efficient recursive reduced-order algorithm for solving (8) is obtained in (Gajic, 1986; Gajic *et al.*, 1990). The algorithm will be briefly summarized here taking into account the specific features of the problem under study.

First of all, we derive expressions for B_c and R_c so that the analogy between the discrete quantities (A, B, Q, R) and continuous ones (A_c, B_c, Q_c, R_c) is completed. By definition

$$S_c^f = B_c^f R_c^{f-1} B_c^{fT}. \quad (14)$$

From (9b) we have

$$S_c^f = 2(I+A)^{-1}S_dD^{-1}(I+A^T)(I+A^T)^{-1}.$$

Since

$$\begin{aligned} S_dD^{-1}(I+A^T) &= S_d[(I+A^T)^{-1}D]^{-1} \\ &= S_d[I + (I+A^T)^{-1}Q(I+A)^{-1}S_d]^{-1} \\ &= B[R + B^T(I+A^T)^{-1}Q(I+A)^{-1}B]^{-1}B^T \end{aligned}$$

[the last step in this expression is justified in (Bar-Ness and Halbersberg, 1980)] we get

$$S_c^f = 2(I+A)^{-1}B[R + B^T(I+A^T)^{-1}Q(I+A)^{-1}B]^{-1} \times B^T(I+A^T)^{-1}. \quad (15)$$

Comparing (16) and (17) we conclude

$$B_c^f = (I+A)^{-1}B = \begin{pmatrix} \varepsilon B_1^f \\ B_2^f \end{pmatrix} \quad (16)$$

and

$$R_c^f = 0.5[R + B^T(I+A^T)^{-1}Q(I+A)^{-1}B] = R_c. \quad (17)$$

Note that R_c is positive definite. The slow time version of (16) is

$$B_c = (1/\varepsilon)B_c^f = \begin{pmatrix} B_1^f \\ B_2^f/\varepsilon \end{pmatrix}. \quad (18)$$

The $O(\varepsilon)$ approximation of (8) subject to (11), (13), and (17)–(18) can be obtained from the following reduced-order set of algebraic equations from (Gajic, 1986; Gajic *et al.*, 1990)

$$0 = \mathbf{P}_1\mathbf{A} + \mathbf{A}^T\mathbf{P}_1 + \mathbf{Q} - \mathbf{P}_1\mathbf{S}\mathbf{P}_1 \quad (19a)$$

$$0 = \mathbf{P}_3A_{22} + A_{22}^T\mathbf{P}_3 + Q_{22} - \mathbf{P}_3S_{22}\mathbf{P}_3 \quad (19b)$$

$$\mathbf{P}_2 = \mathbf{P}_1Z_1 - Z_2 \quad (19c)$$

where newly defined matrices are given by

$$\mathbf{A} = A_0 - B_0R_0^{-1}r^TQ_0, \quad A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21},$$

$$B_0 = B_1 - A_{12}A_{22}^{-1}B_2 \quad R_0 = R_c + r^Tr, \quad r = C_2A_{22}^{-1}B_2,$$

$$\mathbf{S} = B_0R_0^{-1}B_0^T, \quad S_{22} = B_2R_c^{-1}B_2^T \quad Q_0 = C_1 - C_2A_{22}^{-1}A_{21},$$

$$Q_{11} = C_1^TC_1, \quad Q_{22} = C_2^Tc_2, \quad \mathbf{Q} = Q_0^T(I - rR_0^{-1}r^T)Q_0$$

$$Z_1 = (S_{12}\mathbf{P}_3 - A_{12})(A_{22} - S_{22}\mathbf{P}_3)^{-1},$$

$$Z_2 = (A_3^T\mathbf{P}_3 + Q_{12})(A_{22} - S_{22}\mathbf{P}_3)^{-1}.$$

The unique positive semidefinite stabilizing solution of (19) exists under the following assumption.

Assumption. The triples $(\mathbf{A}, B_0, \sqrt{\mathbf{Q}})$ and $(A_{22}, B_2, \sqrt{Q_{22}})$ are stabilizable-detectable.

Defining the approximation errors as

$$P_i = \mathbf{P}_i + \varepsilon E_i, \quad i = 1, 2, 3 \quad (20)$$

the recursive reduced-order algorithm, with the rate of convergence of $O(\varepsilon)$, was derived in Gajic (1986) and Gajic *et al.*, (1990)

$$E_1^{(j+1)}D_1 + D_1^{\mathrm{T}}E_1^{(j+1)} = D^{\mathrm{T}}H_1^{(j)\mathrm{T}} + H_1^{(j)}D + D^{\mathrm{T}}H_3^{(j)}D + \varepsilon H_2^{(j)} \quad (21a)$$

$$E_2^{(j+1)}D_3 + E_1^{(j+1)}D_{21} + D_{22}^{\mathrm{T}}E_3^{(j+1)} = H_1^{(j,j+1)} \quad (21b)$$

$$E_3^{(j+1)}D_3 + D_3^{\mathrm{T}}E_3^{(j+1)} = H_3^{(j)} \quad (21c)$$

with $j = 0, 1, 2, \ldots$, and $E_1^{(0)} = 0$, $E_2^{(0)} = 0$, $E_3^{(0)} = 0$, where newly defined matrices are given by

$$D_1 = A_{11} - S_{11}\mathbf{P}_1 - S_{12}\mathbf{P}_2^{\mathrm{T}} - D_{21}D_3^{-1}D_{22} = D_{11} - D_{21}D_3^{-1}D_{22}$$

$$D_3 = A_{22} - S_{22}\mathbf{P}_3, \qquad D = D_3^{-1}D_{22}$$

$$D_{21} = A_{12} - S_{12}\mathbf{P}_3, \quad D_{22} = A_{21} - S_{12}^{\mathrm{T}}\mathbf{P}_1 - S_{22}\mathbf{P}_2^{\mathrm{T}}$$

$$H_1^{(j,j+1)} = A_{11}^{\mathrm{T}}P_2^{(j)} - P_1^{(j+1)}S_{11}P_2^{(j)} - P_2^{(j)}S_{12}^{\mathrm{T}}P_2^{(j)} - \varepsilon(E_1^{(j+1)}S_{12}E_3^{(j+1)} + E_2^{(j)}S_{22}E_2^{(j)})$$

$$H_2^{(j)} = E_1^{(j)}S_{11}E_1^{(j)} + E_1^{(j)}S_{12}E_2^{(j)\mathrm{T}} + E_2^{(j)}S_{12}^{\mathrm{T}}E_1^{(j)} + E_2^{(j)}S_{22}E_2^{(j)\mathrm{T}}$$

$$H_3^{(j)} = -P_2^{(j)\mathrm{T}}A_{12} - A_{12}^{\mathrm{T}}P_2^{(j)} + \varepsilon P_2^{(j)\mathrm{T}}S_{11}P_2^{(j)} + \varepsilon E_3^{(j)}S_{22}E_3^{(j)} + P_2^{(j)\mathrm{T}}S_{12}P_3^{(j)} + P_3^{(j)}S_{12}^{\mathrm{T}}P_2^{(j)}.$$

It is important to point out that D_1 and D_3 are stable matrices (Gajic, 1986).

The rate of convergence of (21) is $O(\varepsilon)$, that is

$$\|P_i - P_i^{(j)}\| = O(\varepsilon^j), \quad i = 1, 2, 3; \quad j = 0, 1, 2, \ldots \quad (22)$$

where

$$P_i^{(j)} = \mathbf{P}_i + \varepsilon E_i^{(j)}, \quad i = 1, 2, 3; \quad j = 0, 1, 2, \ldots \quad (23)$$

The proposed algorithm for the reduced-order solution of singularly perturbed discrete algebraic Riccati equation has the following form:

(i) Transform (4) into (8) by using the bilinear transformation defined in (9).

(ii) Solve (8) by using the recursive reduced-order algorithm defined by (19)–(23).

Equations (19)–(23) can be implemented by using ideas from parallel programming [(21a) and (21c) can be solved simultaneously] which will play an important role in the real time control.

3. *Near-optimal control of singularly perturbed discrete systems*

The linear-quadratic optimal control problem (1)–(3) has been studied in the context of singular perturbations in (Litkouhi and Khalil, 1984), where the fast time version has been adopted, so that (1) is multiplied by a small perturbation parameter, that is

$$J_f = \varepsilon J. \quad (24)$$

It is proven in (Litkouhi and Khalil, 1985) that the near-optimal control given by

$$u^{(j)}(k) = -(R + B^{\mathrm{T}}P^{(j)}B)^{-1}B^{\mathrm{T}}P^{(j)}Ax(k) = -F^{(j)}x(k) \quad (25)$$

where $P^{(j)}$ satisfies

$$P^{(j)} - P_{\mathrm{opt}} = O(\varepsilon^j) \quad (26)$$

is near-optimal in the sense

$$J_f^{(j)} - J_f^{\mathrm{opt}} = O(\varepsilon^{2j}). \quad (27)$$

The approximate performance $J^{(j)}$ can be obtained from the discrete algebraic Lyapunov equation

$$K^{(j)} = (A - BF^{(j)})^{\mathrm{T}}K^{(j)}(A - BF^{(j)}) + Q + F^{(j)\mathrm{T}}RF^{(j)} \quad (28)$$

so that

$$J^{(j)} = x(0)^{\mathrm{T}}K^{(j)}x(0). \quad (29)$$

In the previous section we have developed a very efficient technique for generating $P^{(j)}$ by using the recursive reduced-order scheme (19)–(21), such that each iteration improves the accuracy by an order of magnitude [see (22)]. Thus, the proposed algorithm from Section 2 and the theoretical results obtained in (Litkouhi and Khalil, 1984) and given in (25)–(27) comprise a new method for solving the linear-quadratic optimal control problem of singularly perturbed discrete systems. The efficiency of this method is demonstrated on a real world example in the next section.

Table 1. Reduced-order solution of the discrete algebraic Riccati equation

j	P1		P2		P3	
0	1.98280	0.00737	−0.36273	−0.30229	0.61994	0.03821
		2.21170	−1.95280	0.37275		0.34262
1	2.12850	−0.00249	−0.40041	−0.31446	0.67749	0.02734
		2.34370	−2.07120	0.40444		0.34488
2	2.13680	−0.00359	−0.40292	−0.31533	0.68128	0.02656
		2.35100	−2.07690	0.40631		0.34514
3	2.13700	−0.00368	−0.40304	−0.31537	0.68147	0.02652
		2.35110	−2.07700	0.40637		0.34516
4	2.13700	−0.00368	−0.40304	−0.31537	0.68147	0.02651
		2.35110	−2.07700	0.40637		0.34516
*	2.13700	−0.00368	−0.40304	−0.31537	0.68147	0.02651
		2.35110	−2.07700	0.40637		0.34516

* = exact solution.

4. *Numerical example*

A linearized model of an F-8 aircraft is considered in Elliott (1977). By a proper scaling this model was presented in the singularly perturbed continuous form (fast time version) (Litkouhi, 1983), with two fast and two slow variables, where $\varepsilon = 1/30$.

The model is discretized in Litkouhi (1983) by using the sampling period $T = 1$, leading to

$$A = \begin{pmatrix} 0.98475 & -0.079903 & 0.0009054 & -0.0010765 \\ 0.041588 & 0.99899 & -0.035855 & 0.012684 \\ -0.54662 & 0.044916 & -0.32991 & 0.19318 \\ 2.6624 & -0.10045 & -0.92455 & -0.26325 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.0037112 & 0.00073610 \\ -0.087051 & 0.0000093411 \\ -1.19844 & -0.00041378 \\ -3.1927 & 0.00092535 \end{pmatrix}.$$

The linear-quadratic optimal control problem is solved for weighting matrices $R = I_2$, $Q = 10^{-2}I_4$ and the initial condition $x(0) = [1, 0, 0.008, 0]^{\mathrm{T}}$.

The eigenvalues of the matrix A_4 are $-0.297 \pm j0.442$, so that equation (6) is satisfied.

Simulation results for the discrete algebraic Riccati equation are presented in Table 1. The approximate values of the criterion are given in Table 2.

Table 2. Near-optimality of the approximate criterion

j	$J_{\mathrm{apr}}^{(j)} - J_{\mathrm{opt}}$
0	0.208×10^{-2}
1	0.885×10^{-5}
2	0.155×10^{-7}
3	0.534×10^{-10}

5. Conclusions

The discrete linear-quadratic control problem is completely and efficiently solved (up to an arbitrary order of accuracy) in terms of the reduced-order problems. The proposed algorithm has a parallel structure and is very convenient for a real time implementation. The approach based on a bilinear transformation (used in this paper) might produce insights in so far unsolved problems in the theory of linear discrete singularly perturbed systems, such as discrete multimodeling, discrete LQ stochastic control and discrete stochastic multimodeling. Solutions to the corresponding continuous time problems are already documented in the literature.

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