OPTIMAL CONTROL APPLICATIONS & METHODS, VOL. 11, 345–354 (1990)

APPROXIMATE PARALLEL CONTROLLERS FOR DISCRETE STOCHASTIC WEAKLY COUPLED LINEAR SYSTEMS

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SUMMARY

The global Kalman filter of linear weakly coupled discrete systems is exactly decomposed into separate reduced-order local filters via the use of a decoupling transformation. The approximate parallel controllers, up to an arbitrary degree of accuracy, are derived by approximating coefficients of the optimal control law. The proposed method allows parallel processing of information and reduces both off-line and on-line computational requirements. A real-world example demonstrates the efficiency of the proposed method.

KEY WORDS Large-scale systems Perturbation techniques System order reduction Stochastic control

1. INTRODUCTION

Linear weakly coupled systems have been studied in different set-ups by many researchers.[1–15] The recursive approach to weakly coupled continuous systems, based on fixed-point iterations, has been developed recently.[10–14] It has been shown that recursive methods are particularly useful when a coupling parameter ε is not extremely small and/or when any desired order of accuracy is required, namely $O(\varepsilon^k)$, where $k = 2, 3, 4, \ldots$.[12] The linear quadratic Gaussian (LQG) control problem of weakly coupled discrete systems has not been studied in the literature. This is due to the fact that the partitioned form of the main equation of the optimal linear control theory, the Riccati equation, has a very complicated form in the discrete time domain.

An efficient algorithm for the solution of discrete weakly coupled algebraic Riccati equations has been obtained recently.[15] The method produces the reduced-order near-optimal solution, up to any arbitrary degree of accuracy, and reduces the number of required off-line computations.

The weakly coupled structure of the global Kalman filter is exploited in this paper such that it may be replaced by two lower-order local filters which will produce additional on-line savings in required computations. This has been achieved via the use of a decoupling transformation[13] which produces the exact block diagonalization of the global filter. The approximate feedback control law is obtained by approximating coefficients of the optimal local filters and regulator gains with an accuracy $O(\varepsilon^N)$. The resulting feedback control law is shown to be a near-optimal solution of the LQG problem by studying the corresponding closed-loop system as a system driven by white noise. It is shown that the order of approximation of the optimal

0143–2087/90/040345–10$05.00

Received 15 January 1990
Revised 7 August 1990

performance is $O(\varepsilon^N)$ and the order of approximation of the optimal system trajectories is $O(\varepsilon^{2N})$. All required coefficients of desired accuracy are obtained by using recursive reduced-order fixed-point-type numerical techniques.[13,15] The obtained numerical algorithms converge to the required optimal coefficients with the rate of convergence $O(\varepsilon^2)$. Corresponding results for another class of small-parameter problems, singularly perturbed systems, have also been obtained recently.[16]

A real-world example, a fifth-order distillation column problem is presented in order to demonstrate the efficiency of the proposed method. Numerical results are included.

2. LINEAR QUADRATIC GAUSSIAN CONTROL OF DISCRETE WEAKLY COUPLED SYSTEMS AT STEADY STATE

Consider the discrete linear weakly coupled stochastic system

$$\mathbf{x}_1(n+1) = \mathbf{A}_{11}\mathbf{x}_1(n) + \varepsilon\mathbf{A}_{12}\mathbf{x}_2(n) + \mathbf{B}_{11}\mathbf{u}_1(n) + \varepsilon\mathbf{B}_{12}\mathbf{u}_2(n) + \mathbf{G}_{11}\mathbf{w}_1(n) + \varepsilon\mathbf{G}_{12}\mathbf{w}_2(n) \quad (1a)$$

$$\mathbf{x}_2(n+1) = \varepsilon\mathbf{A}_{21}\mathbf{x}_1(n) + \mathbf{A}_{22}\mathbf{x}_2(n) + \varepsilon\mathbf{B}_{21}\mathbf{u}_1(n) + \mathbf{B}_{22}\mathbf{u}_2(n) + \varepsilon\mathbf{G}_{21}\mathbf{w}_1(n) + \mathbf{G}_{22}\mathbf{w}_2(n) \quad (1b)$$

$$\mathbf{y}_1(n) = \mathbf{C}_{11}\mathbf{x}_1(n) + \varepsilon\mathbf{C}_{12}\mathbf{x}_2(n) + \mathbf{v}_1(n) \quad (2a)$$

$$\mathbf{y}_2(n) = \varepsilon\mathbf{C}_{21}\mathbf{x}_1(n) + \mathbf{C}_{22}\mathbf{x}_2(n) + \mathbf{v}_2(n) \quad (2b)$$

with the performance criterion

$$J = \mathrm{E}\left(\sum_{n=0}^{\infty} [\mathbf{z}^{\mathrm{T}}(n)\mathbf{z}(n) + \mathbf{u}_1^{\mathrm{T}}(n)\mathbf{R}_1\mathbf{u}_1(n) + \mathbf{u}_2^{\mathrm{T}}(n)\mathbf{R}_2\mathbf{u}_2(n)]\right) \quad (3)$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$ are the state vectors, $\mathbf{u}_i \in \mathbb{R}^{m_i}$ are control inputs, $\mathbf{y}_i \in \mathbb{R}^{l_i}$ are observed outputs, $\mathbf{w}_i \in \mathbb{R}^{r_i}$ and $\mathbf{v}_i \in \mathbb{R}^{l_i}$ are independent zero-mean stationary Gaussian mutually uncorrelated white noise processes with intensities $\mathbf{W}_i > 0$ and $\mathbf{V}_i > 0$ respectively and $\mathbf{z}_i \in \mathbb{R}^{s_i}$, $i = 1, 2$, are controlled outputs given by

$$\mathbf{z}_1(n) = \mathbf{D}_{11}\mathbf{x}_1(n) + \varepsilon\mathbf{D}_{12}\mathbf{x}_2(n) \quad (4a)$$

$$\mathbf{z}_2(n) = \varepsilon\mathbf{D}_{21}\mathbf{x}_1(n) + \mathbf{D}_{22}\mathbf{x}_2(n) \quad (4b)$$

All matrices ($\mathbf{A}_{ij}$, $\mathbf{B}_{ij}$, $\mathbf{C}_{ij}$, $\mathbf{D}_{ij}$ and $\mathbf{G}_{ij}$, $ij = 11, 12, 21, 22$) are bounded functions of a small coupling parameter ε and have appropriate dimensions.[10-12] In addition, it is assumed that $\mathbf{R}_i$, $i = 1, 2$, are positive definite matrices.

The optimal control law is given by[17]

$$\mathbf{u}(n) = -\mathbf{F}\hat{\mathbf{x}}(n) \quad (5)$$

with

$$\hat{\mathbf{x}}(n+1) = \mathbf{A}\hat{\mathbf{x}}(n) + \mathbf{B}\mathbf{u}(n) + \mathbf{K}[\mathbf{y}(n) - \mathbf{C}\hat{\mathbf{x}}(n)] \quad (6)$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \varepsilon\mathbf{A}_{12} \\ \varepsilon\mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_{11} & \varepsilon\mathbf{B}_{12} \\ \varepsilon\mathbf{B}_{21} & \mathbf{B}_{22} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_{11} & \varepsilon\mathbf{C}_{12} \\ \varepsilon\mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{11} & \varepsilon\mathbf{K}_{12} \\ \varepsilon\mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} \mathbf{F}_{11} & \varepsilon\mathbf{F}_{12} \\ \varepsilon\mathbf{F}_{21} & \mathbf{F}_{22} \end{bmatrix}$$

The regulator and filter gains **F** and **K** are obtained from

$$\mathbf{F} = (\mathbf{R} + \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{A} \tag{7}$$

$$\mathbf{K} = \mathbf{A}\mathbf{Q}\mathbf{C}^{\mathrm{T}}(\mathbf{V} + \mathbf{C}\mathbf{Q}\mathbf{C}^{\mathrm{T}})^{-1} \tag{8}$$

where **P** and **Q** are positive semidefinite stabilizing solutions of the discrete time algebraic regulator and filter Riccati equations respectively, given by

$$\mathbf{P} = \mathbf{D}^{\mathrm{T}}\mathbf{D} + \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{A} - \mathbf{A}^{\mathrm{T}}\mathbf{P}\mathbf{B}(\mathbf{R} + \mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{B})^{-1}\mathbf{B}^{\mathrm{T}}\mathbf{P}\mathbf{A} \tag{9}$$

$$\mathbf{Q} = \mathbf{A}\mathbf{Q}\mathbf{A}^{\mathrm{T}} - \mathbf{A}\mathbf{Q}\mathbf{C}^{\mathrm{T}}(\mathbf{V} + \mathbf{C}\mathbf{Q}\mathbf{C}^{\mathrm{T}})^{-1}\mathbf{C}\mathbf{Q}\mathbf{A}^{\mathrm{T}} + \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}} \tag{10}$$

with

$$\mathbf{R} = \operatorname{diag}(\mathbf{R}_1\ \mathbf{R}_2), \qquad \mathbf{W} = \operatorname{diag}(\mathbf{W}_1\ \mathbf{W}_2), \qquad \mathbf{V} = \operatorname{diag}(\mathbf{V}_1\ \mathbf{V}_2)$$

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \varepsilon\mathbf{D}_{12} \\ \varepsilon\mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix}, \qquad \mathbf{G} = \begin{bmatrix} \mathbf{G}_{11} & \varepsilon\mathbf{G}_{12} \\ \varepsilon\mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix}$$

Owing to the block-dominant structure of the problem matrices, the required solutions **P** and **Q** have the form

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \varepsilon\mathbf{P}_{12} \\ \varepsilon\mathbf{P}_{12}^{\mathrm{T}} & \mathbf{P}_{22} \end{bmatrix}, \qquad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \varepsilon\mathbf{Q}_{12} \\ \varepsilon\mathbf{Q}_{12}^{\mathrm{T}} & \mathbf{Q}_{22} \end{bmatrix} \tag{11}$$

In order to obtain the required solutions of (9) and (10) in terms of the reduced-order problems and to overcome the complicated partitioned form of the discrete time algebraic Riccati equation, we have used the method developed in Reference 15 (which is based on a bilinear transformation[18]) to transform the discrete Riccati equations (9) and (10) into continuous time algebraic Riccati equations of the form

$$\mathbf{A}_{\mathrm{R}}^{\mathrm{T}}P + P\mathbf{A}_{\mathrm{R}} - P\mathbf{S}_{\mathrm{R}}P + \mathbf{D}_{\mathrm{R}}^{\mathrm{T}}\mathbf{D}_{\mathrm{R}} = 0, \qquad \mathbf{S}_{\mathrm{R}} = \mathbf{B}_{\mathrm{R}}\mathbf{R}_{\mathrm{R}}^{-1}\mathbf{B}_{\mathrm{R}}^{\mathrm{T}} \tag{12}$$

$$\mathbf{A}_{\mathrm{F}}Q + Q\mathbf{A}_{\mathrm{F}}^{\mathrm{T}} - Q\mathbf{S}_{\mathrm{F}}Q + \mathbf{G}_{\mathrm{F}}\mathbf{W}_{\mathrm{F}}\mathbf{G}_{\mathrm{F}}^{\mathrm{T}} = 0, \qquad \mathbf{S}_{\mathrm{F}} = \mathbf{C}_{\mathrm{F}}^{\mathrm{T}}\mathbf{V}_{\mathrm{F}}^{-1}\mathbf{C}_{\mathrm{F}} \tag{13}$$

such that solutions of (9) and (10) are equal to the solutions of (12) and (13); that is,

$$\mathbf{P} = P, \qquad \mathbf{Q} = Q \tag{14}$$

where

$$\begin{aligned} \mathbf{A}_{\mathrm{R}} &= \mathbf{I} - 2(\boldsymbol{\Delta}_{\mathrm{R}}^{-1})^{\mathrm{T}} \\ \mathbf{B}_{\mathrm{R}}\mathbf{R}_{\mathrm{R}}^{-1}\mathbf{B}_{\mathrm{R}}^{\mathrm{T}} &= 2(\mathbf{I} + \mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}}\boldsymbol{\Delta}_{\mathrm{R}}^{-1} \\ \mathbf{D}_{\mathrm{R}}^{\mathrm{T}}\mathbf{D}_{\mathrm{R}} &= 2\boldsymbol{\Delta}_{\mathrm{R}}^{-1}\mathbf{D}^{\mathrm{T}}\mathbf{D}(\mathbf{I} + \mathbf{A})^{-1} \\ \boldsymbol{\Delta}_{\mathrm{R}} &= (\mathbf{I} + \mathbf{A}^{\mathrm{T}}) + \mathbf{D}^{\mathrm{T}}\mathbf{D}(\mathbf{I} + \mathbf{A})^{-1}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^{\mathrm{T}} \end{aligned} \tag{15a}$$

and

$$\begin{aligned} \mathbf{A}_{\mathrm{F}} &= \mathbf{I} - 2(\boldsymbol{\Delta}_{\mathrm{F}}^{-1}) \\ \mathbf{C}_{\mathrm{F}}^{\mathrm{T}}\mathbf{V}_{\mathrm{F}}^{-1}\mathbf{C}_{\mathrm{F}} &= 2(\mathbf{I} + \mathbf{A}^{\mathrm{T}})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{C}\boldsymbol{\Delta}_{\mathrm{F}}^{-1} \\ \mathbf{G}_{\mathrm{F}}\mathbf{W}_{\mathrm{F}}\mathbf{G}_{\mathrm{F}}^{\mathrm{T}} &= 2\boldsymbol{\Delta}_{\mathrm{F}}^{-1}\mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}(\mathbf{I} + \mathbf{A}^{\mathrm{T}})^{-1} \\ \boldsymbol{\Delta}_{\mathrm{F}} &= (\mathbf{I} + \mathbf{A}) + \mathbf{G}\mathbf{W}\mathbf{G}^{\mathrm{T}}(\mathbf{I} + \mathbf{A}^{\mathrm{T}})^{-1}\mathbf{C}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{C} \end{aligned} \tag{15b}$$

Equations (12) and (13) preserve the structure of weakly coupled systems.[15] These equations can be solved in terms of reduced-order problems very efficiently by using the recursive method developed in Reference 12, which converges with the rate of convergence $O(\varepsilon^2)$. The method is briefly summarized in Appendix I. Solutions of (12) and (13) are found in terms of reduced-order problems by imposing standard stabilizability–detectability assumptions on the subsystems (see Appendix I).

Getting approximate solutions for **P** and **Q** in terms of the reduced-order problems will produce savings in off-line computations. However, in the case of stochastic systems, where an additional dynamical system or filter has to be built, one is particularly interested in the reduction of on-line computations. In this paper that will be achieved by using a decoupling transformation.[13] The basic properties of that transformation and its discrete time version are given in Appendix II.

The Kalman filter (6) is viewed as a system driven by the innovation process. However, one might study the filter form when it is driven by both measurements and controls. The filter form under consideration is obtained from (6) as

$$\hat{\mathbf{x}}_1(n+1) = (\mathbf{A}_{11} - \mathbf{B}_{11}\mathbf{F}_{11} - \varepsilon^2\mathbf{B}_{12}\mathbf{F}_{21})\hat{\mathbf{x}}_1(n) + \varepsilon(\mathbf{A}_{12} - \mathbf{B}_{11}\mathbf{F}_{12} - \mathbf{B}_{12}\mathbf{F}_{22})\hat{\mathbf{x}}_2(n) + \mathbf{K}_{11}\boldsymbol{\nu}_1(n) + \varepsilon\mathbf{K}_{12}\boldsymbol{\nu}_2(n) \quad (16a)$$

$$\hat{\mathbf{x}}_2(n+1) = \varepsilon(\mathbf{A}_{21} - \mathbf{B}_{21}\mathbf{F}_{11} - \mathbf{B}_{22}\mathbf{F}_{21})\hat{\mathbf{x}}_1(n) + (\mathbf{A}_{22} - \varepsilon^2\mathbf{B}_{21}\mathbf{F}_{12} - \mathbf{B}_{22}\mathbf{F}_{22})\hat{\mathbf{x}}_2(n) + \varepsilon\mathbf{K}_{21}\boldsymbol{\nu}_1(n) + \mathbf{K}_{22}\boldsymbol{\nu}_2(n) \quad (16b)$$

with innovation processes

$$\boldsymbol{\nu}_1(n) = \mathbf{y}_1(n) - \mathbf{C}_{11}\hat{\mathbf{x}}_1(n) - \varepsilon\mathbf{C}_{12}\hat{\mathbf{x}}_2(n)$$
$$\boldsymbol{\nu}_2(n) = \mathbf{y}_2(n) - \varepsilon\mathbf{C}_{21}\hat{\mathbf{x}}_1(n) - \mathbf{C}_{22}\hat{\mathbf{x}}_2(n)$$

The non-singular state transformation of Reference 13 diagonalizes (16) under the condition that $\mathbf{A}_{11} - \mathbf{B}_{11}\mathbf{F}_{11} - \varepsilon^2\mathbf{B}_{12}\mathbf{F}_{21}$ and $\mathbf{A}_{22} - \mathbf{B}_{22}\mathbf{F}_{22} - \varepsilon^2\mathbf{B}_{21}\mathbf{F}_{12}$ have no eigenvalues in common (see Appendix II). The transformation is given by

$$\begin{bmatrix}\hat{\boldsymbol{\eta}}_1(n)\\ \hat{\boldsymbol{\eta}}_2(n)\end{bmatrix} = \begin{bmatrix}\mathbf{I} - \varepsilon^2\mathbf{LH} & -\varepsilon\mathbf{L}\\ \varepsilon\mathbf{H} & \mathbf{I}\end{bmatrix}\begin{bmatrix}\hat{\mathbf{x}}_1(n)\\ \hat{\mathbf{x}}_2(n)\end{bmatrix} = \mathbf{T}\begin{bmatrix}\hat{\mathbf{x}}_1(n)\\ \hat{\mathbf{x}}_2(n)\end{bmatrix} \quad (17)$$

with

$$\mathbf{T}^{-1} = \begin{bmatrix}\mathbf{I} & \varepsilon\mathbf{L}\\ -\varepsilon\mathbf{H} & \mathbf{I} - \varepsilon^2\mathbf{HL}\end{bmatrix} \quad (18)$$

where matrices **L** and **H** satisfy equations

$$\mathbf{H}\mathbf{a}_{11} - \mathbf{a}_{22}\mathbf{H} + \mathbf{a}_{21} - \varepsilon^2\mathbf{H}\mathbf{a}_{12}\mathbf{H} = 0 \quad (19)$$

$$\mathbf{L}(\mathbf{a}_{22} + \varepsilon\mathbf{H}\mathbf{a}_{12}) - (\mathbf{a}_{11} - \varepsilon^2\mathbf{a}_{12}\mathbf{H})\mathbf{L} - \mathbf{a}_{12} = 0 \quad (20)$$

with

$$\mathbf{a}_{11} = \mathbf{A}_{11} - \mathbf{B}_{11}\mathbf{F}_{11} - \varepsilon^2\mathbf{B}_{12}\mathbf{F}_{12}$$
$$\mathbf{a}_{12} = \mathbf{A}_{12} - \mathbf{B}_{11}\mathbf{F}_{12} - \mathbf{B}_{12}\mathbf{F}_{22}$$
$$\mathbf{a}_{21} = \mathbf{A}_{21} - \mathbf{B}_{21}\mathbf{F}_{11} - \mathbf{B}_{22}\mathbf{F}_{21}$$
$$\mathbf{a}_{22} = \mathbf{A}_{22} - \mathbf{B}_{22}\mathbf{F}_{22} - \varepsilon^2\mathbf{B}_{21}\mathbf{F}_{12}$$

The optimal feedback control, expressed in the new co-ordinates, has the form

$$\mathbf{u}_1(n) = -\mathbf{f}_{11}\hat{\boldsymbol{\eta}}_1(n) - \varepsilon\mathbf{f}_{12}\hat{\boldsymbol{\eta}}_2(n) \quad (21a)$$

$$\mathbf{u}_2(n) = -\varepsilon\mathbf{f}_{21}\hat{\boldsymbol{\eta}}_1(n) - \mathbf{f}_{22}\hat{\boldsymbol{\eta}}_2(n) \quad (21b)$$

with

$$\hat{\boldsymbol{\eta}}_1(n+1) = \boldsymbol{\alpha}_1\hat{\boldsymbol{\eta}}_1(n) + \boldsymbol{\beta}_{11}\boldsymbol{\nu}_1(n) - \varepsilon\boldsymbol{\beta}_{12}\boldsymbol{\nu}_2(n) \quad (22a)$$

$$\hat{\boldsymbol{\eta}}_2(n+1) = \boldsymbol{\alpha}_1\hat{\boldsymbol{\eta}}_2(n) + \varepsilon\boldsymbol{\beta}_{21}\boldsymbol{\nu}_1(n) + \boldsymbol{\beta}_{22}\boldsymbol{\nu}_2(n) \quad (22b)$$

where

$$\mathbf{f}_{11} = \mathbf{F}_{11} - \varepsilon^2\mathbf{F}_{12}\mathbf{H}, \qquad \mathbf{f}_{12} = \mathbf{F}_{12} + (\mathbf{F}_{11} - \varepsilon^2\mathbf{F}_{12}\mathbf{H})\mathbf{L}$$

$$\mathbf{f}_{21} = \mathbf{F}_{21} - \mathbf{F}_{22}\mathbf{H}, \qquad \mathbf{f}_{22} = \mathbf{F}_{22} + \varepsilon^2(\mathbf{F}_{21} - \mathbf{F}_{22}\mathbf{H})\mathbf{L}$$

$$\boldsymbol{\alpha}_1 = \mathbf{a}_{11} - \varepsilon^2\mathbf{a}_{12}\mathbf{H}, \qquad \boldsymbol{\alpha}_2 = \mathbf{a}_{22} + \varepsilon^2\mathbf{H}\mathbf{a}_{12}$$

$$\boldsymbol{\beta}_{11} = \mathbf{K}_{11} - \varepsilon^2\mathbf{L}(\mathbf{H} + \mathbf{K}_{21}), \qquad \boldsymbol{\beta}_{12} = \mathbf{K}_{12} - \mathbf{L}\mathbf{K}_{22} - \varepsilon^2\mathbf{L}\mathbf{H}\mathbf{K}_{12}$$

$$\boldsymbol{\beta}_{21} = \mathbf{H}\mathbf{K}_{11} + \mathbf{K}_{21}, \qquad \boldsymbol{\beta}_{22} = \mathbf{K}_{22} + \varepsilon^2\mathbf{H}\mathbf{K}_{12}$$

The innovation processes $\boldsymbol{\nu}_1$ and $\boldsymbol{\nu}_2$ are now given by

$$\boldsymbol{\nu}_1(n) = \mathbf{y}_1(n) - \mathbf{d}_{11}\hat{\boldsymbol{\eta}}_1(n) - \varepsilon\mathbf{d}_{12}\hat{\boldsymbol{\eta}}_2(n) \quad (23a)$$

$$\boldsymbol{\nu}_2(n) = \mathbf{y}_2(n) - \varepsilon\mathbf{d}_{21}\hat{\boldsymbol{\eta}}_1(n) - \mathbf{d}_{22}\hat{\boldsymbol{\eta}}_2(n) \quad (23b)$$

where

$$\mathbf{d}_{11} = \mathbf{C}_{11} - \varepsilon^2\mathbf{C}_{12}\mathbf{H}, \qquad \mathbf{d}_{12} = \mathbf{C}_{11}\mathbf{L} + \mathbf{C}_{12} - \varepsilon^2\mathbf{C}_{12}\mathbf{H}\mathbf{L}$$

$$\mathbf{d}_{21} = \mathbf{C}_{21} - \mathbf{C}_{22}\mathbf{H}, \qquad \mathbf{d}_{22} = \mathbf{C}_{22} + \varepsilon^2(\mathbf{C}_{21} - \mathbf{C}_{22}\mathbf{H})\mathbf{L}$$

Approximate control laws are defined by perturbing coefficients $\mathbf{F}_{ij}$, $\mathbf{K}_{ij}$ $(i, j = 1, 2)$, $\mathbf{L}$ and $\mathbf{H}$ by $O(\varepsilon^k)$, $k = 1, 2, \ldots$, in other words by using kth approximations for these coefficients, where k stands for the required order of accuracy. Thus we write

$$\mathbf{u}_1^{(k)}(n) = -\mathbf{f}_{11}^{(k)}\hat{\boldsymbol{\eta}}_1^{(k)}(n) - \varepsilon\mathbf{f}_{12}^{(k)}\hat{\boldsymbol{\eta}}_2^{(k)}(n) \quad (24a)$$

$$\mathbf{u}_2^{(k)}(n) = -\varepsilon\mathbf{f}_{21}^{(k)}\hat{\boldsymbol{\eta}}_1^{(k)}(n) - \mathbf{f}_{22}^{(k)}\hat{\boldsymbol{\eta}}_2^{(k)}(n) \quad (24b)$$

with

$$\hat{\boldsymbol{\eta}}_1^{(k)}(n+1) = \boldsymbol{\alpha}_1^{(k)}\hat{\boldsymbol{\eta}}_1^{(k)}(n) + \boldsymbol{\beta}_{11}^{(k)}\boldsymbol{\nu}_1^{(k)}(n) + \varepsilon\boldsymbol{\beta}_{12}^{(k)}\boldsymbol{\nu}_2^{(k)}(n) \quad (25a)$$

$$\hat{\boldsymbol{\eta}}_2^{(k)}(n+1) = \boldsymbol{\alpha}_2^{(k)}\hat{\boldsymbol{\eta}}_2^{(k)}(n) + \varepsilon\boldsymbol{\beta}_{21}^{(k)}\boldsymbol{\nu}_1^{(k)}(n) + \boldsymbol{\beta}_{22}^{(k)}\boldsymbol{\nu}_2^{(k)}(n) \quad (25b)$$

where

$$\boldsymbol{\nu}_1^{(k)}(n) = \mathbf{y}_1(n) - \mathbf{d}_{11}^{(k)}\hat{\boldsymbol{\eta}}_1^{(k)}(n) - \varepsilon\mathbf{d}_{12}^{(k)}\hat{\boldsymbol{\eta}}_2^{(k)}(n) \quad (26a)$$

$$\boldsymbol{\nu}_2^{(k)}(n) = \mathbf{y}_2(n) - \varepsilon\mathbf{d}_{21}^{(k)}\hat{\boldsymbol{\eta}}_1^{(k)}(n) - \mathbf{d}_{22}^{(k)}\hat{\boldsymbol{\eta}}_2^{(k)}(n) \quad (26b)$$

and

$$\mathbf{f}_{ij}^{(k)} = \mathbf{f}_{ij} + O(\varepsilon^k), \qquad \mathbf{d}_{ij}^{(k)} = \mathbf{d}_{ij} + O(\varepsilon^k)$$

$$\boldsymbol{\beta}_{ij}^{(k)} = \boldsymbol{\beta}_{ij} + O(\varepsilon^k), \qquad \boldsymbol{\alpha}_{ij}^{(k)} = \boldsymbol{\alpha}_{ij} + O(\varepsilon^k)$$

$(i, j = 1, 2)$.

The approximate values of $J^{(k)}$ are obtained from the equations

$$J^{(k)} = \mathrm{E}\left(\sum_{n=0}^{\infty} [\mathbf{x}^{(k)^{\mathrm{T}}}(n)\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{x}^{(k)}(n) + \mathbf{u}^{(k)^{\mathrm{T}}}(n)\mathbf{R}\mathbf{u}^{(k)}(n)]\right) = \mathrm{tr}(\mathbf{D}^{\mathrm{T}}\mathbf{D}\mathbf{q}_{11}^{(k)} + \mathbf{f}^{(k)^{\mathrm{T}}}\mathbf{R}\mathbf{f}^{(k)}\mathbf{q}_{22}^{(k)}) \quad (27)$$

where

$$\mathbf{q}_{11}^{(k)} = \mathrm{var}(\mathbf{x}_1^{(k)}\ \mathbf{x}_2^{(k)})^{\mathrm{T}}, \qquad \mathbf{q}_{22}^{(k)} = \mathrm{var}(\hat{\boldsymbol{\eta}}_1^{(k)}\ \hat{\boldsymbol{\eta}}_2^{(k)})^{\mathrm{T}}$$

$$\mathbf{u}^{(k)} = \begin{bmatrix} \mathbf{u}_1^{(k)}(n) \\ \mathbf{u}_2^{(k)}(n) \end{bmatrix}, \qquad \mathbf{f}^{(k)} = \begin{bmatrix} \mathbf{f}_{11}^{(k)} & \varepsilon\mathbf{f}_{12}^{(k)} \\ \varepsilon\mathbf{f}_{21}^{(k)} & \mathbf{f}_{22}^{(k)} \end{bmatrix}$$

The quantities $\mathbf{q}_{11}^{(k)}$ and $\mathbf{q}_{22}^{(k)}$ can be obtained by studying the variance equation of the following system driven by white noise:

$$\begin{bmatrix} \mathbf{x}^{(k)}(n+1) \\ \hat{\boldsymbol{\eta}}^{(k)}(n+1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{f}^{(k)} \\ \boldsymbol{\beta}^{(k)}\mathbf{C} & \boldsymbol{\alpha}^{(k)} - \boldsymbol{\beta}^{(k)}\mathbf{d}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{x}^{(k)}(n) \\ \hat{\boldsymbol{\eta}}^{(k)}(n) \end{bmatrix} + \begin{bmatrix} \mathbf{G} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\beta}^{(k)} \end{bmatrix} \begin{bmatrix} \mathbf{w}(n) \\ \mathbf{v}(n) \end{bmatrix} \tag{28}$$

where

$$\boldsymbol{\alpha}^{(k)} = \begin{bmatrix} \alpha_1^{(k)} & \mathbf{0} \\ \mathbf{0} & \alpha_2^{(k)} \end{bmatrix}, \qquad \boldsymbol{\beta}^{(k)} = \begin{bmatrix} \beta_{11}^{(k)} & \varepsilon\beta_{12}^{(k)} \\ \varepsilon\beta_{21}^{(k)} & \beta_{22}^{(k)} \end{bmatrix}, \qquad \mathbf{d}^{(k)} = \begin{bmatrix} \mathbf{d}_{11}^{(k)} & \varepsilon\mathbf{d}_{12}^{(k)} \\ \varepsilon\mathbf{d}_{21}^{(k)} & \mathbf{d}_{22}^{(k)} \end{bmatrix}$$

Equation (28) can be represented in the composite form

$$\boldsymbol{\Gamma}^{(k)}(n+1) = \boldsymbol{\Lambda}^{(k)}\boldsymbol{\Gamma}^{(k)}(n) + \boldsymbol{\Pi}^{(k)}\boldsymbol{\omega}(n) \tag{29}$$

with obvious definitions for $\boldsymbol{\Lambda}^{(k)}$, $\boldsymbol{\Pi}^{(k)}$, $\boldsymbol{\Gamma}^{(k)}(n)$ and $\boldsymbol{\omega}(n)$. The steady state variance of $\boldsymbol{\Gamma}^{(k)}(n)$ is denoted by $\mathbf{q}^{(k)}$ and is given by the discrete algebraic Lyapunov equation[17]

$$\mathbf{q}^{(k)} = \boldsymbol{\Lambda}^{(k)}\mathbf{q}^{(k)}\boldsymbol{\Lambda}^{(k)\mathrm{T}} + \boldsymbol{\Pi}^{(k)}W\boldsymbol{\Pi}^{(k)\mathrm{T}}, \qquad W = \mathrm{diag}(\mathbf{W}\ \ \mathbf{V}) \tag{30}$$

with $\mathbf{q}^{(k)}$ partitioned as

$$\mathbf{q}^{(k)} = \begin{bmatrix} \mathbf{q}_{11}^{(k)} & \mathbf{q}_{12}^{(k)} \\ \mathbf{q}_{12}^{(k)\mathrm{T}} & \mathbf{q}_{22}^{(k)} \end{bmatrix} \tag{31}$$

The optimal value of J has the very well known form[17]

$$J^{\mathrm{opt}} = \mathrm{tr}\,[\mathbf{D}^{\mathrm{T}}\mathbf{D}Q + P\mathbf{K}(\mathbf{C}Q\mathbf{C}^{\mathrm{T}} + \mathbf{V})\mathbf{K}^{\mathrm{T}}] \tag{32}$$

where P, Q, $\mathbf{F}$ and $\mathbf{K}$ are obtained from (7)–(10).

The near optimality of the proposed approximate control law (24) is established in the following theorem.

Theorem 1

Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the optimal trajectories and let J be the optimal value of the performance criterion. Let $\mathbf{x}_1^{(k)}$, $\mathbf{x}_2^{(k)}$ and $J^{(k)}$ be the corresponding quantities under the approximate control law $\mathbf{u}^{(k)}$ given by (24). Then, under the conditions stated in Assumption 1 (Appendix I) and Lemma 1 (Appendix II), the following hold

$$J^{\mathrm{opt}} - J^{(k)} = O(\varepsilon^k) \tag{33a}$$

$$\mathrm{var}(\mathbf{x}_i - \mathbf{x}_i^{(k)}) = O(\varepsilon^{2k}), \quad k = 0, 1, 2, \ldots \tag{33b}$$

The proof of this theorem is rather lengthy and is therefore omitted here. It follows the ideas of Theorems 1 and 2 from Reference 19 obtained for another class of small-parameter problems, singularly perturbed systems. These two theorems have been proved in the context of weakly coupled linear systems.[14] In addition, owing to the discrete nature of the problem, the proof of our Theorem 1 utilizes a bilinear transformation[20] which transforms the discrete Lyapunov equation (30) into a continuous one and compares it with the corresponding equation under the optimal control law.[12,21]

3. NUMERICAL EXAMPLE

Here we treat a real-world physical example, a fifth-order distillation column problem,[22] in order to demonstrate the efficiency of the proposed method. The problem matrices **A** and **B** are given by

$$\mathbf{A} = 10^{-3}\begin{bmatrix} 989\cdot50 & 5\cdot6382 & 0\cdot2589 & 0\cdot0125 & 0\cdot0006 \\ 117\cdot25 & 814\cdot50 & 76\cdot038 & 5\cdot5526 & 0\cdot3700 \\ 8\cdot7680 & 123\cdot87 & 750\cdot20 & 107\cdot96 & 11\cdot245 \\ 0\cdot9108 & 17\cdot991 & 183\cdot81 & 668\cdot34 & 150\cdot78 \\ 0\cdot0179 & 0\cdot3172 & 1\cdot6974 & 13\cdot298 & 985\cdot19 \end{bmatrix}$$

$$\mathbf{B}^{\mathrm{T}} = 10^{-3}\begin{bmatrix} 0\cdot0192 & 6\cdot0733 & 8\cdot2911 & 9\cdot1965 & 0\cdot7025 \\ -0\cdot0013 & -0\cdot6192 & -13\cdot339 & -18\cdot442 & -1\cdot4252 \end{bmatrix}$$

where dim $\mathbf{A}_{11} = 2 \times 2$ and dim $\mathbf{A}_{22} = 3 \times 3$. The remaining matrices are chosen as

$$\mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mathbf{Q} = \mathbf{I}_5, \qquad \mathbf{R} = \mathbf{I}_2$$

We assume that $\mathbf{G} = \mathbf{B}$ and that the white noise intensity matrices are given by

$$W_1 = 1, \qquad W_2 = 1, \qquad V_1 = 0\cdot1, \qquad V_2 = 0\cdot1$$

Matrices **A** and **B** are obtained from Reference 22 by performing a discretization with the sampling rate $\Delta T = 0\cdot1$. The small weakly coupling parameter ε is built into the problem. It can be roughly estimated from the strongest coupled matrix, in this case matrix **B**. Apparently, the strongest coupling is in the third row, i.e.

$$\varepsilon = \frac{b_{31}}{b_{32}} = \frac{8\cdot2911}{13\cdot339} \approx 0\cdot62$$

In practice, how the problem matrices are partitioned will determine the choice of the coupling parameter which in turn determines the rate of convergence and the domain of attraction of the iterative scheme to the optimal solution. It is desirable to get as small as possible a value of the small coupling parameter. This will speed up the convergence process. However, the small coupling parameter is built into the problem and one cannot go beyond the physical limits. The choice of the value of the small coupling parameter is also discussed in Reference 21.

Simulation results, obtained by using the software package L-A-S,[23] are presented in Table I. It can be seen that despite the relatively large value of the coupling parameter ε, we have very rapid convergence to the optimal solution.

Table 1. Approximate values for the performance criterion

k	$J^{(k)}$	$J^{(k)} - J$
0	$0\cdot80528 \times 10^{-2}$	$0\cdot6989 \times 10^{-3}$
1	$0\cdot75977 \times 10^{-2}$	$0\cdot2438 \times 10^{-3}$
2	$0\cdot74277 \times 10^{-2}$	$0\cdot7380 \times 10^{-4}$
4	$0\cdot73887 \times 10^{-2}$	$0\cdot3480 \times 10^{-4}$
6	$0\cdot73546 \times 10^{-2}$	$0\cdot5000 \times 10^{-6}$
8	$0\cdot73539 \times 10^{-2}$	$< 10^{-7}$
Optimal	$0\cdot73539 \times 10^{-2}$	

4. CONCLUSIONS

Near-optimum (up to any desired accuracy) steady state regulators are obtained for stochastic linear weakly coupled discrete systems. The proposed method reduces considerably the number of required off-line and on-line computations since it introduces full parallelism in the design procedure.

APPENDIX I

Consider the continuous time algebraic Riccati equation for weakly coupled systems:

$$\mathbf{PA} + \mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{Q} - \mathbf{PSP} = 0 \tag{34}$$

where

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \varepsilon\mathbf{P}_2 \\ \varepsilon\mathbf{P}_2^{\mathrm{T}} & \mathbf{P}_3 \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \varepsilon\mathbf{A}_2 \\ \varepsilon\mathbf{A}_3 & \mathbf{A}_4 \end{bmatrix}$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_1 & \varepsilon\mathbf{Q}_2 \\ \varepsilon\mathbf{Q}_2^{\mathrm{T}} & \mathbf{Q}_3 \end{bmatrix}, \qquad \mathbf{S} = \begin{bmatrix} \mathbf{S}_1 & \varepsilon\mathbf{S}_2 \\ \varepsilon\mathbf{S}_2^{\mathrm{T}} & \mathbf{S}_3 \end{bmatrix}$$

Partitioning (34) and setting $\varepsilon = 0$ will produce

$$\underline{\mathbf{P}}_1\mathbf{A}_1 + \mathbf{A}_1^{\mathrm{T}}\underline{\mathbf{P}}_1 + \mathbf{Q}_1 - \underline{\mathbf{P}}_1\mathbf{S}_1\underline{\mathbf{P}}_1 = 0 \tag{35}$$

$$\underline{\mathbf{P}}_3\mathbf{A}_4 + \mathbf{A}_4^{\mathrm{T}}\underline{\mathbf{P}}_3 + \mathbf{Q}_3 - \underline{\mathbf{P}}_3\mathbf{S}_3\underline{\mathbf{P}}_3 = 0 \tag{36}$$

$$\underline{\mathbf{P}}_2(\mathbf{A}_4 - \mathbf{S}_3\underline{\mathbf{P}}_3) + (\mathbf{A}_1 - \mathbf{S}_1\underline{\mathbf{P}}_1)^{\mathrm{T}}\underline{\mathbf{P}}_2 + \underline{\mathbf{P}}_1\mathbf{A}_2 + \mathbf{A}_3^{\mathrm{T}}\underline{\mathbf{P}}_3 + \mathbf{Q}_2 - \underline{\mathbf{P}}_1\mathbf{S}_2\underline{\mathbf{P}}_3 = 0 \tag{37}$$

The unique positive semi-definite stabilizing solution of (35)–(37) exists under the following assumption.

Assumption 1

Triples $(\mathbf{A}_1, \sqrt{\mathbf{S}_1}, \sqrt{\mathbf{Q}_1})$ and $(\mathbf{A}_4, \sqrt{\mathbf{S}_3}, \sqrt{\mathbf{Q}_3})$ are stabilizable–detectable.

Solutions of (34) and (35)–(37) are related by[12]

$$\mathbf{P}_i = \underline{\mathbf{P}}_i + \varepsilon^2\mathbf{E}_i, \quad i = 1, 2, 3 \tag{38}$$

where the $\mathbf{E}_i$, $i = 1, 2, 3$, are obtained from a reduced-order recursive algorithm of the form

$$\mathbf{E}_1^{(j+1)}\mathbf{\Delta}_1 + \mathbf{\Delta}_1^{\mathrm{T}}\mathbf{E}_1^{(j+1)} = \mathbf{M}_1^{(j)} \tag{39}$$

$$\mathbf{E}_3^{(j+1)}\mathbf{\Delta}_2 + \mathbf{\Delta}_2^{\mathrm{T}}\mathbf{E}_3^{(j+1)} = \mathbf{M}_3^{(j)} \tag{40}$$

$$\mathbf{E}_2^{(j+1)}\mathbf{\Delta}_2 + \mathbf{\Delta}_1^{\mathrm{T}}\mathbf{E}_2^{(j+1)} + \mathbf{E}_1^{(j+1)}\mathbf{\Delta}_{12} + \mathbf{\Delta}_{21}\mathbf{E}_3^{(j+1)} = \mathbf{M}_2^{(j,j+1)} \tag{41}$$

with $j = 0, 1, 2, \ldots$ and $\mathbf{E}_1^{(0)} = 0$, $\mathbf{E}_2^{(0)} = 0$, $\mathbf{E}_3^{(0)} = 0$. Newly defined matrices are given by

$$\begin{aligned}
&\mathbf{\Delta}_1 = \mathbf{A}_1 - \mathbf{S}_1\underline{\mathbf{P}}_1, \qquad \mathbf{\Delta}_2 = \mathbf{A}_4 - \mathbf{S}_3\underline{\mathbf{P}}_3 \\
&\mathbf{\Delta}_{12} = \mathbf{A}_2 - \mathbf{S}_1\underline{\mathbf{P}}_2 - \mathbf{S}_2\underline{\mathbf{P}}_3, \qquad \mathbf{\Delta}_{21} = \mathbf{A}_3 - \mathbf{S}_3\underline{\mathbf{P}}_2^{\mathrm{T}} - \mathbf{S}_2^{\mathrm{T}}\underline{\mathbf{P}}_1 \\
&\mathbf{M}_1^{(j)} = \mathbf{P}_1^{(j)}\mathbf{S}_2\mathbf{P}_1^{(j)} + \mathbf{P}_2^{(j)}\mathbf{S}_2^{\mathrm{T}}\mathbf{P}_1^{(j)} + \mathbf{P}_1^{(j)}\mathbf{S}_2\mathbf{P}_2^{(j)\mathrm{T}} - \mathbf{P}_2^{(j)}\mathbf{A}_3 + \mathbf{P}_2^{(j)}\mathbf{S}_3\mathbf{P}_2^{(j)\mathrm{T}} \\
&\qquad - \varepsilon^2\mathbf{E}_1^{(j)}\mathbf{S}_1\mathbf{E}_1^{(j)} - \mathbf{A}_3\mathbf{P}_2^{(j)\mathrm{T}} \\
&\mathbf{M}_3^{(j)} = \mathbf{P}_3^{(j)}\mathbf{S}_2^{\mathrm{T}}\mathbf{P}_3^{(j)} + \mathbf{P}_2^{(j)\mathrm{T}}\mathbf{S}_1\mathbf{P}_2^{(j)} + \mathbf{P}_3^{(j)}\mathbf{S}_2^{\mathrm{T}}\mathbf{P}_2^{(j)} + \mathbf{P}_2^{(j)\mathrm{T}}\mathbf{S}_2\mathbf{P}_3^{(j)} + \varepsilon^2\mathbf{E}_3^{(j)}\mathbf{S}_3\mathbf{E}_3^{(j)} \\
&\qquad - \mathbf{P}_2^{(j)\mathrm{T}}\mathbf{A}_2 - \mathbf{A}_2^{\mathrm{T}}\mathbf{P}_2^{(j)} \\
&\mathbf{M}_2^{(j,j+1)} = \mathbf{P}_2^{(j)}\mathbf{S}_2^{\mathrm{T}}\mathbf{P}_2^{(j)} + \varepsilon^2(\mathbf{E}_1^{(j+1)}\mathbf{S}_2\mathbf{E}_3^{(j+1)} + \mathbf{E}_1^{(j+1)}\mathbf{S}_1\mathbf{E}_2^{(j)} + \mathbf{E}_2^{(j)}\mathbf{S}_3\mathbf{E}_3^{(j+1)})
\end{aligned} \tag{42}$$

It is important to point out that Δ_1 and Δ_2 are stable matrices.[12] The rate of convergence of (39)–(41) is $O(\varepsilon^2)$, i.e.[12]

$$\| \mathbf{P}_i - \mathbf{P}_i^{(j)} \| = O(\varepsilon^{2j}), \quad i = 1, 2, 3, \quad j = 0, 1, 2, \ldots \tag{43}$$

where

$$\mathbf{P}_i^{(j)} = \underline{\mathbf{P}}_i + \varepsilon^2 \mathbf{E}_i^{(j)}, \quad i = 1, 2, 3, \quad j = 0, 1, 2, \ldots$$

APPENDIX II

Consider the weakly coupled linear discrete system

$$\begin{aligned} \mathbf{x}_1(n+1) &= \mathbf{A}_1\mathbf{x}_1(n) + \varepsilon\mathbf{A}_2\mathbf{x}_2(n) \\ \mathbf{x}_2(n+1) &= \varepsilon\mathbf{A}_3\mathbf{x}_1(n) + \mathbf{A}_4\mathbf{x}_2(n) \end{aligned} \tag{44}$$

By forming the discrete version of Reference 13 we can show that the non-singular transformation

$$\mathbf{T} = \begin{bmatrix} \mathbf{I} - \varepsilon^2\mathbf{LH} & -\varepsilon\mathbf{L} \\ \varepsilon\mathbf{H} & \mathbf{I} \end{bmatrix}, \qquad \mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I} & \varepsilon\mathbf{L} \\ -\varepsilon\mathbf{H} & \mathbf{I} - \varepsilon^2\mathbf{HL} \end{bmatrix} \tag{45}$$

transforms (44) into the block diagonal form

$$\begin{aligned} \boldsymbol{\eta}_1(n+1) &= (\mathbf{A}_1 - \varepsilon^2\mathbf{A}_2\mathbf{H})\boldsymbol{\eta}_1(n) \\ \boldsymbol{\eta}_2(n+1) &= (\mathbf{A}_4 - \varepsilon^2\mathbf{HA}_2)\boldsymbol{\eta}_2(n) \end{aligned} \tag{46}$$

The transformation matrices $\mathbf{L}$ and $\mathbf{H}$ satisfy algebraic equations

$$\mathbf{HA}_1 - \mathbf{A}_4\mathbf{H} + \mathbf{A}_3 - \varepsilon^2\mathbf{HA}_2\mathbf{H} = 0 \tag{47}$$

$$\mathbf{L}(\mathbf{A}_4 + \varepsilon^2\mathbf{HA}_2) - (\mathbf{A}_1 - \varepsilon^2\mathbf{A}_2\mathbf{H})\mathbf{L} - \mathbf{A}_2 = 0 \tag{48}$$

The following lemma is obtained in Reference 13.

Lemma 1

Under the assumption that the matrices $\mathbf{A}_1$ and $\mathbf{A}_4$ have no eigenvalues in common, there exists a small parameter ε such that unique solutions of (47) and (48) exist.

In our problem the matrices $\mathbf{A}_1$ and $\mathbf{A}_4$ are feedback matrices so that the assumption of Lemma 1 is almost always satisfied.

The numerical solution of (47) can be obtained by using a fixed-point-type algorithm[13]

$$\mathbf{H}^{(j+1)}\mathbf{A}_1 - \mathbf{A}_4\mathbf{H}^{(j+1)} + \mathbf{A}_3 - \varepsilon^2\mathbf{H}^{(j)}\mathbf{A}_2\mathbf{H}^{(j)} = 0, \quad j = 0, 1, 2, \ldots, N-1 \tag{49}$$

where $\mathbf{H}^{(0)}$ is obtained from

$$\mathbf{H}^{(0)}\mathbf{A}_1 - \mathbf{A}_4\mathbf{H}^{(0)} + \mathbf{A}_3 = 0$$

This algorithm has the rate of convergence $O(\varepsilon^2)$.[13] Having obtained $\mathbf{H}^{(N)}$, we can solve equation (48) as a linear Sylvester-type equation. Another way of solving (47) exploits the Newton method.[13]

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