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Brief Paper

Near-optimum Steady State Regulators for Stochastic Linear Weakly Coupled Systems*

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Key Words—Large scale systems; perturbation techniques; system order reduction; Kalman filters; stochastic control.

Abstract—This paper presents an approach to the decomposition and approximation of the linear quadratic Gaussian estimation and control problems for weakly coupled systems. The global Kalman filter is decomposed into separate reduced-order local filters via the use of a decoupling transformation. A near-optimal control law is derived by approximating the coefficients of the truly optimal control law. The order of approximation of the optimal performance is $O(\varepsilon^N)$, where N is the order of approximation of the coefficients. A real world power system example demonstrates the failure of $O(\varepsilon^2)$ and $O(\varepsilon^4)$ approximations and the necessity for the existence of the $O(\varepsilon^N)$ theory. The proposed method produces the reduction in both off-line and on-line computational requirements and leads to convergence under mild assumptions. In addition, only low-order systems are involved in algebraic calculations and no analyticity requirement (a standard assumption for the power series method) is imposed on system coefficients.

1. *Introduction*

LINEAR WEAKLY coupled systems have been studied in different setups by many researchers (Kokotovic *et al.*, 1969; Delacour *et al.*, 1978; Petkovski and Rakic, 1979; Mahmoud, 1978; Petrovic and Gajic, 1988; Harkara *et al.*, 1989; Gajic *et al.*, 1990; Sezer and Siljak, 1986; Ishimatsu *et al.*, 1975; Washburn and Mendel, 1980; Khalil and Kokotovic, 1978). The solution of the main equation of linear optimal control theory for weakly coupled systems—the Riccati equation—has been obtained in terms of a power series expansion of a small coupling parameter ε (Kokotovic *et al.*, 1969). Approximate feedback control laws have been derived by truncating expansions of the feedback coefficients of the optimal control law (Kokotovic *et al.*, 1969; Delacour *et al.*, 1978; Gajic *et al.*, 1990; Khalil and Kokotovic, 1978). Such approximations have been shown to be near-optimal with performance which can be made as close to optimal as desired by including a sufficient number of terms in the truncated expansions. The recursive approach to weakly coupled systems, based on the fixed point iterations, has been developed recently (Petrovic and Gajic, 1988; Harkara *et al.*, 1989; Gajic *et al.*, 1990). It has been shown that the recursive methods are particularly useful when a coupling parameter ε is not extremely small and/or when any desired order of accuracy is required, namely, $O(\varepsilon^k)$,§ where $k = 2, 3, 4, \ldots$, (Gajic *et al.*, 1990).

The linear quadratic Gaussian control problem of weakly coupled systems has not been studied in the literature. A corresponding result for another class of small parameter systems—singularly perturbed systems (Kokotovic and Khalil, 1986)—is obtained in Khalil and Gajic (1984)—from the power series expansion point of view; and in (Gajic, 1986), by using the fixed point theory.

The recursive solution of the algebraic regulator and filter Riccati equations, and corresponding approximations to the regulator and filter gains in terms of reduced-order problems, were obtained in Gajic *et al.* (1990). Such approximations are not sufficient because they only reduce the off-line computations (Gajic *et al.*, 1990), and cannot help with the on-line computations of implementing the Kalman filter which will be of the same order as the overall weakly coupled system. The weakly coupled structure of the global filter is exploited in this paper such that it may be replaced by two lower-order local filters. This has been achieved via the use of a decoupling transformation introduced in Gajic and Shen (1989).

The paper presents an approach to the decomposition and approximation of the linear quadratic Gaussian (LQG) control problem of weakly coupled systems by treating the decomposition and approximation tasks separately from each other. The decoupling transformation (Gajic and Shen, 1989) is used for the exact block diagonalization of the global Kalman filter. The approximate feedback control law is then obtained by approximating the coefficients of the optimal local filters with the accuracy of $O(\varepsilon^N)$. The resulting feedback control law is shown to be a near optimal solution of the LQG problem by studying the corresponding closed loop system as a system driven by white noise. It is shown that the order of approximation of the optimal performance is $O(\varepsilon^N)$, and the order of approximation of the optimal system trajectories is $O(\varepsilon^{2N})$. All required coefficients of desired accuracy are easily obtained by using the recursive fixed point type numerical techniques developed in Gajic *et al.* (1990) and Gajic and Shen (1989). The numerical algorithms given in the paper converge to the required coefficients with the rate of convergence of $O(\varepsilon^2)$. In addition, only low-order subsystems are involved in algebraic computations and no analyticity requirement is imposed on system coefficients—which is the standard assumption in the power series expansion method (Petrovic and Gajic, 1988; Gajic *et al.*, 1990). As a consequence of these properties, under very mild conditions [coefficients are bounded functions of a small coupling parameter over a compact set $\varepsilon \in [0, \varepsilon_1]$, (Petrovic and Gajic, 1988; Zangwill and Garcia, 1981), in addition to the standard stabilizability-detectability subsystem assumptions], we have achieved the reduction in both off-line and on-line computational requirements.

The efficiency of the proposed method is demonstrated on a power system composed of two interconnected areas (Geromel and Peres, 1985). The small parameter ε is relatively big in this example, that is $\varepsilon = 0.7$. Since $O(0.7^{26}) \approx 10^{-4}$, it will require 24 terms in order to get the accuracy of 10^{-4} if the power series expansion method has been used—which is not feasible. On the other hand, the fixed point method scheme used in this paper will demand 12

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§ $O(\varepsilon^k)$ stand for $C\varepsilon^k$, where C is a bounded constant and k is any arbitrary constant.

iterations [rate of convergence is $O(\varepsilon^2)$] of the presented algorithms— which can be done relatively easily. Even more, it happens in this problem that the $O(\varepsilon^2)$ and $O(\varepsilon^4)$ approximate filters do not stabilize the plant-filter augmented system, so that the approximate filter has to be found with the accuracy of at least $O(\varepsilon^6)$.

2. *Approximation of weakly coupled systems driven by white noise*

Consider the linear time-invariant weakly coupled system driven by white noise

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}A_{11}(\varepsilon) & \varepsilon A_{12}(\varepsilon)\\ \varepsilon A_{21}(\varepsilon) & A_{22}(\varepsilon)\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}G_{11}(\varepsilon) & \varepsilon G_{12}(\varepsilon)\\ \varepsilon G_{21}(\varepsilon) & G_{22}(\varepsilon)\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} \quad (2.1)$$

where $x_i \in \mathcal{R}^{n_i}$, $w_i \in \mathcal{R}^{s_i}$, $i = 1, 2$, and ε is a small parameter. The system matrices are bounded functions of ε, and are of appropriate dimensions. The inputs $w_i(t)$ are zero mean, stationary, Gaussian uncorrelated white noise processes with intensities $W_i > 0$, $i = 1, 2$. It is well known that the variance of the linear system driven by white noise is given by the Lyapunov equation (Kwakernaak and Sivan, 1972). In order to assure the existence of the solution of the Lyapunov equation we assume that $A_{ii}(\varepsilon)$, $i = 1, 2$ are stable matrices.

The purpose of this section is to study approximations of $x_i(t)$, $i = 1, 2$, when ε is small. We are interested in approximations $x_i^N(t)$ which are defined by the following equations

$$\begin{pmatrix}\dot{x}_1^N\\ \dot{x}_2^N\end{pmatrix}=\begin{pmatrix}A_{11}^N(\varepsilon) & \varepsilon A_{12}^N(\varepsilon)\\ \varepsilon A_{21}^N(\varepsilon) & A_{22}^N(\varepsilon)\end{pmatrix}\begin{pmatrix}x_1^N\\ x_2^N\end{pmatrix}+\begin{pmatrix}G_{11}^N(\varepsilon) & \varepsilon G_{12}^N(\varepsilon)\\ \varepsilon G_{21}^N(\varepsilon) & G_{22}^N(\varepsilon)\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} \quad (2.2)$$

where

$$\begin{aligned}A_{ij}(\varepsilon)-A_{ij}^N(\varepsilon)&=O(\varepsilon^N), \quad i,j=1,2.\\ G_{ij}(\varepsilon)-G_{ij}^N(\varepsilon)&=O(\varepsilon^N), \quad i,j=1,2.\end{aligned} \quad (2.3)$$

The quantities of interest are steady-state variances of the errors

$$e_i(t)=x_i(t)-x_i^N(t), \quad i=1,2. \quad (2.4)$$

We study the impact of steady-state errors on a quadratic form

$$\sigma=\operatorname{tr}\left\{\begin{pmatrix}H^{\mathrm{T}}(\varepsilon)H(\varepsilon) & \varepsilon H^{\mathrm{T}}(\varepsilon)J(\varepsilon)\\ \varepsilon J^{\mathrm{T}}(\varepsilon)H(\varepsilon) & J^{\mathrm{T}}(\varepsilon)J(\varepsilon)\end{pmatrix}\times E\begin{pmatrix}x_1(t)x_1^{\mathrm{T}}(t) & x_1(t)x_2^{\mathrm{T}}(t)\\ x_2(t)x_1^{\mathrm{T}}(t) & x_2(t)x_2^{\mathrm{T}}(t)\end{pmatrix}\right\} \quad (2.5)$$

where $H(\varepsilon)$ and $J(\varepsilon)$ are bounded functions of ε. Such a quadratic form appears in the LQG control problem which will be under consideration in the next section. We examine the approximation of σ by σ^N defined by

$$\sigma^N=\operatorname{tr}\left\{\begin{pmatrix}H^{N^{\mathrm{T}}}(\varepsilon)H^N(\varepsilon) & \varepsilon H^{N^{\mathrm{T}}}(\varepsilon)J^N(\varepsilon)\\ \varepsilon J^{N^{\mathrm{T}}}(\varepsilon)H^N(\varepsilon) & J^{N^{\mathrm{T}}}(\varepsilon)J^N(\varepsilon)\end{pmatrix}\times E\begin{pmatrix}x_1^N(t)x_1^{N^{\mathrm{T}}}(t) & x_1^N(t)x_2^{N^{\mathrm{T}}}(t)\\ x_2^N(t)x_1^{N^{\mathrm{T}}}(t) & x_2^N(t)x_2^{N^{\mathrm{T}}}(t)\end{pmatrix}\right\} \quad (2.6)$$

where

$$H^N(\varepsilon)-H(\varepsilon)=O(\varepsilon^N), \quad J^N(\varepsilon)-J(\varepsilon)=O(\varepsilon^N). \quad (2.7)$$

In the following we will suppress the ε-dependence of the problem matrices in order to simplify notation.

The main results of this section are given in the following theorem.

Theorem 1. Under stability assumptions imposed on A_{ii}, $i = 1, 2$; the variances of the errors and the quadratic forms (2.5) and (2.6) at steady state satisfy

$$\operatorname{Var}\{e_i\}=\operatorname{Var}\{x_i-x_i^N\}=O(\varepsilon^{2N}), \quad i=1,2 \quad (2.8a)$$

$$\Delta\sigma=\sigma-\sigma^N=O(\varepsilon^N). \quad (2.8b)$$

A proof for this theorem can be obtained by studying the augmented system composed of (2.2) and the dynamics of (2.4).

Details can be found in Shen (1990) and Gajic *et al.* (1990).

3. *Linear quadratic Gaussian control*

Consider the controlled, weakly coupled linear stochastic system

$$\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix}=\begin{pmatrix}A_{11} & \varepsilon A_{12}\\ \varepsilon A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}B_{11} & \varepsilon B_{12}\\ \varepsilon B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}G_{11} & \varepsilon G_{12}\\ \varepsilon G_{21} & G_{22}\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} \quad (3.1)$$

$$\begin{pmatrix}y_1\\ y_2\end{pmatrix}=\begin{pmatrix}C_{11} & \varepsilon C_{12}\\ \varepsilon C_{21} & C_{22}\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}+\begin{pmatrix}v_1\\ v_2\end{pmatrix} \quad (3.2)$$

where $x_i \in \mathcal{R}^{n_i}$, $u_i \in \mathcal{R}^{m_i}$, $y_i \in \mathcal{R}^{r_i}$, $i = 1, 2$ are state, control and measurement vectors respectively, and $w_i \in \mathcal{R}^{s_i}$, $v_i \in \mathcal{R}^{r_i}$, $i = 1, 2$ are independent zero-mean stationary white Gaussian noise processes with intensities $W_i > 0$, $V_i > 0$, $i = 1, 2$. The degree of interaction between subsystems is measured by a small parameter ε. With (3.1)–(3.2), consider the performance criterion

$$\sigma=\operatorname{tr}\left\{D^{\mathrm{T}}D\cdot E\begin{pmatrix}x_1x_1^{\mathrm{T}} & x_1x_2^{\mathrm{T}}\\ x_2x_1^{\mathrm{T}} & x_2x_2^{\mathrm{T}}\end{pmatrix}+R\cdot E\begin{pmatrix}u_1u_1^{\mathrm{T}} & u_1u_2^{\mathrm{T}}\\ u_2u_1^{\mathrm{T}} & u_2u_2^{\mathrm{T}}\end{pmatrix}\right\} \quad (3.3)$$

which has to be minimized at steady state with positive definite R. In the following all matrices are bounded functions of ε, of appropriate dimensions. In addition, matrices $D^{\mathrm{T}}D$ and R have a weakly coupled structure. We assume that they are given by

$$D^{\mathrm{T}}D=\begin{pmatrix}D_1^{\mathrm{T}}D_1 & \varepsilon D_1^{\mathrm{T}}D_2\\ \varepsilon D_2^{\mathrm{T}}D_1 & D_2^{\mathrm{T}}D_2\end{pmatrix}, \quad R=\begin{pmatrix}R_1 & 0\\ 0 & R_2\end{pmatrix}$$

where $R_i \in \mathcal{R}^{m_i \times m_i}$ and $D_i^{\mathrm{T}}D_i \in \mathcal{R}^{n_i \times n_i}$, $i = 1, 2$.

The optimal control law of (3.1)–(3.3) has a very well-known form (Kwakernaak and Sivan, 1972)

$$\begin{pmatrix}u_1\\ u_2\end{pmatrix}=-\begin{pmatrix}F_{11} & \varepsilon F_{12}\\ \varepsilon F_{21} & F_{22}\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} \quad (3.4)$$

$$\begin{pmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{pmatrix}=\begin{pmatrix}A_{11} & \varepsilon A_{12}\\ \varepsilon A_{21} & A_{22}\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix}+\begin{pmatrix}B_{11} & \varepsilon B_{12}\\ \varepsilon B_{21} & B_{22}\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix}+\begin{pmatrix}K_{11} & \varepsilon K_{12}\\ \varepsilon K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}y_1-C_{11}\hat{x}_1-\varepsilon C_{12}\hat{x}_2\\ y_2-\varepsilon C_{21}\hat{x}_1-C_{22}\hat{x}_2\end{pmatrix}. \quad (3.5)$$

Introducing the notation

$$A=\begin{pmatrix}A_{11} & \varepsilon A_{12}\\ \varepsilon A_{21} & A_{22}\end{pmatrix}, \quad B=\begin{pmatrix}B_{11} & \varepsilon B_{12}\\ \varepsilon B_{21} & B_{22}\end{pmatrix},$$

$$G=\begin{pmatrix}G_{11} & \varepsilon G_{12}\\ \varepsilon G_{21} & G_{22}\end{pmatrix} \quad C=\begin{pmatrix}C_{11} & \varepsilon C_{12}\\ \varepsilon C_{21} & C_{22}\end{pmatrix},$$

$$F=\begin{pmatrix}F_{11} & \varepsilon F_{12}\\ \varepsilon F_{21} & F_{22}\end{pmatrix}, \quad K=\begin{pmatrix}K_{11} & \varepsilon K_{12}\\ \varepsilon K_{21} & K_{22}\end{pmatrix}$$

$$W=\begin{pmatrix}W_1 & 0\\ 0 & W_2\end{pmatrix}, \quad V=\begin{pmatrix}V_1 & 0\\ 0 & V_2\end{pmatrix}$$

the regulator and filter gains are obtained from

$$F=R^{-1}B^{\mathrm{T}}P, \quad K=QC^{\mathrm{T}}V^{-1} \quad (3.6)$$

where P and Q are positive semi-definite stabilizing solutions of algebraic Riccati equations

$$A^{\mathrm{T}}P+PA-PS_RP+D^{\mathrm{T}}D=0, \quad S_R=BR^{-1}B^{\mathrm{T}} \quad (3.7)$$

$$AQ+QA^{\mathrm{T}}-QS_FQ+GWG^{\mathrm{T}}=0, \quad S_F=C^{\mathrm{T}}V^{-1}C. \quad (3.8)$$

Due to the weakly coupled structure of all coefficients in (3.7)–(3.8), solutions of these equations have the form

$$P=\begin{pmatrix}P_1 & \varepsilon P_2\\ \varepsilon P_2^{\mathrm{T}} & P_3\end{pmatrix}, \quad Q=\begin{pmatrix}Q_1 & \varepsilon Q_2\\ \varepsilon Q_2^{\mathrm{T}} & Q_3\end{pmatrix}. \quad (3.9)$$

Solutions of (3.7)–(3.8) were found in terms of reduced-order problems by imposing standard stabilizability-detectability assumptions on subsystems in Gajic *et al.* (1990).

Assumption 1. The triples (A_{ii}, B_{ii}, D_i), $i = 1, 2$ are stabilizable-detectable.

Assumption 2. The triples (A_{ii}, C_{ii}, G_{ii}), $i = 1, 2$ are stabilizable-detectable.

The efficient fixed point reduced-order algorithm [with a rate of convergence of $O(\varepsilon^2)$] for solving (3.7) and (3.8), under Assumptions 1 and 2, was obtained in Petrovic and Gajic (1988) and Gajic *et al.* (1990). Obtaining approximate solutions for P and Q in terms of reduced-order problems will produce savings in off-line computations (Gajic *et al.*, 1990).

In the case of stochastic systems, where an additional dynamical system filter has to be built, one is particularly interested in the reduction of on-line computations. In this paper the reduction of on-line computations will be achieved via the use of a decoupling transformation introduced in Gajic and Shen (1989).

The Kalman filter (3.5) is viewed as a system driven by the measurements and the controls. However, one might study the filter form also with only an "innovations" driving term. Such a filter form is obtained from (3.5) as

$$\begin{pmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{pmatrix} = \begin{pmatrix}(A_{11}-B_{11}F_{11}-\varepsilon^2 B_{12}F_{12}) & \varepsilon(A_{12}-B_{11}F_{12}-B_{12}F_{22})\\ \varepsilon(A_{21}-B_{21}F_{11}-B_{22}F_{21}) & (A_{22}-B_{22}F_{22}-\varepsilon^2 B_{21}F_{12})\end{pmatrix} \times \begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} + \begin{pmatrix}K_{11} & \varepsilon K_{12}\\ \varepsilon K_{21} & K_{22}\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} \quad (3.10)$$

with the innovation processes given by

$$v_1 = y_1 - C_{11}\hat{x}_1 - \varepsilon C_{12}\hat{x}_2 \quad (3.11a)$$

$$v_2 = y_2 - \varepsilon C_{21}\hat{x}_1 - C_{22}\hat{x}. \quad (3.11b)$$

The nonsingular state transformation of Gajic and Shen (1989) will block diagonalize (3.10) under condition that $(A_{11} - B_{11}F_{11} - \varepsilon^2 B_{12}F_{21})$ and $(A_{22} - B_{22}F_{22} - \varepsilon^2 B_{21}F_{12})$ have no eigenvalues in common. This transformation is given by

$$\begin{pmatrix}\hat{\eta}_1\\ \hat{\eta}_2\end{pmatrix} = \begin{pmatrix}I-\varepsilon^2 LH & -\varepsilon L\\ \varepsilon H & I\end{pmatrix}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} = T^{-1}\begin{pmatrix}\hat{x}_1\\ \hat{x}_2\end{pmatrix} \quad (3.12)$$

where

$$T = \begin{pmatrix}I & \varepsilon L\\ -\varepsilon H & I-\varepsilon^2 HL\end{pmatrix} \quad (3.13)$$

with L and H matrices satisfying algebraic equations given in Gajic and Shen (1989). The optimal feedback control expressed in the new coordinates has the form

$$u_1 = -f_{11}\hat{\eta}_1 - \varepsilon f_{12}\hat{\eta}_2 \quad (3.14a)$$

$$u_2 = -\varepsilon f_{21}\hat{\eta}_1 - f_{22}\hat{\eta}_2 \quad (3.14b)$$

with

$$\dot{\hat{\eta}}_1 = \alpha_1\hat{\eta}_1 + \beta_{11}v_1 + \varepsilon\beta_{12}v_2 \quad (3.15a)$$

$$\dot{\hat{\eta}}_2 = \alpha_2\hat{\eta}_2 + \varepsilon\beta_{21}v_1 + \beta_{22}v_2 \quad (3.15b)$$

where

$$\begin{gathered} f_{11} = F_{11} - \varepsilon^2 F_{12}H, \quad f_{12} = F_{12} + (F_{11} - \varepsilon^2 F_{12}H)L\\ f_{21} = F_{21} - F_{22}H, \quad f_{22} = F_{22} + \varepsilon^2(F_{21} - F_{22}H)L\\ \alpha_1 = a_{11} - \varepsilon^2 a_{12}H, \quad \alpha_2 = a_{22} + \varepsilon^2 H a_{12}\\ \beta_{11} = K_{11} - \varepsilon^2(LH + LK_{21}), \quad \beta_{12} = K_{12} - LK_{22} - \varepsilon^2 LHK_{12}\\ \beta_{21} = HK_{11} + K_{21}, \quad \beta_{22} = K_{22} + \varepsilon^2 HK_{12} \end{gathered} \quad (3.16)$$

and

$$\begin{aligned} a_{11} &= (A_{11} - B_{11}F_{11} - \varepsilon^2 B_{12}F_{21}),\\ a_{12} &= (A_{12} - B_{11}F_{12} - B_{12}F_{22})\\ a_{21} &= (A_{21} - B_{21}F_{11} - B_{22}F_{21}),\\ a_{22} &= (A_{22} - B_{22}F_{22} - \varepsilon^2 B_{21}F_{12}). \end{aligned}$$

The innovation processes v_1 and v_2 in the new coordinates are given by

$$v_1 = y_1 - d_{11}\hat{\eta}_1 - \varepsilon d_{12}\hat{\eta}_2 \quad (3.17a)$$

$$v_2 = y_2 - \varepsilon d_{21}\hat{\eta}_1 - d_{22}\hat{\eta}_2 \quad (3.17b)$$

where

$$\begin{aligned} d_{11} &= C_{11} - \varepsilon^2 C_{12}H, \quad d_{12} = C_{11}L + C_{12} - \varepsilon^2 C_{12}HL\\ d_{21} &= C_{21} - C_{22}H, \quad d_{22} = C_{22} + \varepsilon^2(C_{21} - C_{22}H)L. \end{aligned} \quad (3.18)$$

Approximate control laws are defined by perturbing coefficients F_{ij}, K_{ij} $(i, j = 1, 2)$, L and H by $O(\varepsilon^k)$, $k = 1, 2, \ldots$, in other words by using kth order approximations for these coefficients, where k stands for the required order of accuracy, that is

$$u_1^{(k)} = -f_{11}^{(k)}\hat{\eta}_1^{(k)} - \varepsilon f_{12}^{(k)}\hat{\eta}_2^{(k)} \quad (3.19a)$$

$$u_2^{(k)} = -\varepsilon f_{21}^{(k)}\hat{\eta}_1^{(k)} - f_{22}^{(k)}\hat{\eta}_2^{(k)} \quad (3.19b)$$

with

$$\dot{\hat{\eta}}_1^{(k)} = \alpha_1^{(k)}\hat{\eta}_1^{(k)} + \beta_{11}^{(k)}v_1^{(k)} + \varepsilon\beta_{12}^{(k)}v_2^{(k)} \quad (3.20a)$$

$$\dot{\hat{\eta}}_2^{(k)} = \alpha_2^{(k)}\hat{\eta}_2^{(k)} + \varepsilon\beta_{21}^{(k)}v_1^{(k)} + \beta_{22}^{(k)}v_2^{(k)} \quad (3.20b)$$

where

$$v_1^{(k)} = y_1 - d_{11}^{(k)}\hat{\eta}_1^{(k)} - \varepsilon d_{12}^{(k)}\hat{\eta}_2^{(k)} \quad (3.21a)$$

$$v_2^{(k)} = y_2 - \varepsilon d_{21}^{(k)}\hat{\eta}_1^{(k)} - d_{22}^{(k)}\hat{\eta}_2^{(k)} \quad (3.21b)$$

and

$$\begin{aligned} f_{ij}^{(k)} &= f_{ij} + O(\varepsilon^k); \quad d_{ij}^{(k)} = d_{ij} + O(\varepsilon^k)\\ \beta_{ij}^{(k)} &= \beta_{ij} + O(\varepsilon^k); \quad \alpha_{ij}^{(k)} = \alpha_{ij} + O(\varepsilon^k) \quad i, j = 1, 2. \end{aligned} \quad (3.22)$$

The near-optimality of the proposed control law (3.19) is established in the following theorem.

Theorem 2. Let x_1 and x_2 be the optimal trajectories and σ be the optimal value of the performance criterion. Let $x_1^{(k)}$, $x_2^{(k)}$ and $\sigma^{(k)}$ be the corresponding quantities under the approximate control law $u^{(k)}$, then

$$\sigma - \sigma^{(k)} = O(\varepsilon^k), \quad k = 0, 1, 2, \ldots \quad (3.23)$$

$$\mathrm{Var}\,\{(x_i - x_i^{(k)})\} = O(\varepsilon^{2k}), \quad k = 0, 1, 2, \ldots. \quad (3.24)$$

Proof of Theorem 2. The result of Section 2 is employed by studying systems of equations driven by white noise. For the truly optimal control, consider the equations

$$\begin{pmatrix}\dot{\hat{\eta}}_1\\ \dot{e}_1\\ \dot{\hat{\eta}}_2\\ \dot{e}_2\end{pmatrix} = \begin{pmatrix}\Lambda_{11} & \varepsilon\Lambda_{12}\\ \varepsilon\Lambda_{21} & \Lambda_{22}\end{pmatrix}\begin{pmatrix}\hat{\eta}_1\\ e_1\\ \hat{\eta}_2\\ e_2\end{pmatrix} + \begin{pmatrix}\mathcal{B}_{11} & \varepsilon\mathcal{B}_{12}\\ \varepsilon\mathcal{B}_{21} & \mathcal{B}_{22}\end{pmatrix}\begin{pmatrix}v_1\\ w_1\\ v_2\\ w_2\end{pmatrix} \quad (3.25)$$

where $e_i = \eta_i - \hat{\eta}_i$, $i = 1, 2$ are the estimation errors. Equations for the approximate control are

$$\begin{pmatrix}\dot{\hat{\eta}}_1^N\\ \dot{e}_1^N\\ \dot{\hat{\eta}}_2^N\\ \dot{e}_2^N\end{pmatrix} = \begin{pmatrix}\Lambda_{11}^N & \varepsilon\Lambda_{12}^N\\ \varepsilon\Lambda_{21}^N & \Lambda_{22}^N\end{pmatrix}\begin{pmatrix}\hat{\eta}_1^N\\ e_1^N\\ \hat{\eta}_2^N\\ e_2^N\end{pmatrix} + \begin{pmatrix}\mathcal{B}_{11}^N & \varepsilon\mathcal{B}_{12}^N\\ \varepsilon\mathcal{B}_{21}^N & \mathcal{B}_{22}^N\end{pmatrix}\begin{pmatrix}v_1\\ w_1\\ v_2\\ w_2\end{pmatrix} \quad (3.26)$$

where $e_i^N = \eta_i - \hat{\eta}_i^N$ are corresponding estimation errors. Matrices Λ_{ij}, $\mathcal{B}_{ij}$ and Λ_{ij}^N, $\mathcal{B}_{ij}^N$ in (3.25) and (3.26) are obtained in an obvious way. It can be verified that

$$\Lambda_{ij} - \Lambda_{ij}^N = O(\varepsilon^N); \quad \mathcal{B}_{ij} - \mathcal{B}_{ij}^N = O(\varepsilon^N), \quad i, j = 1, 2$$

and that $\Lambda_{ii}(0)$, $i = 1, 2$ are given by

$$\Lambda_{ii}(0) = \begin{pmatrix}A_{ii} - B_{ii}F_{ii} & K_{ii}C_{ii}\\ 0 & A_{ii} - K_{ii}C_{ii}\end{pmatrix} \quad (3.27)$$

which by Assumptions 1 *and* 2 imposed on triples (A_{ii}, B_{ii}, D_i) and (A_{ii}, C_{ii}, G_{ii}), $i = 1, 2$ guarantee the stability of matrices $\Lambda_{ii}(0)$. The results of Theorem 1 given by (2.8) can be now directly used to establish (3.23) and (3.24).

Results obtained in Theorem 2 are along the lines of those reported in Khalil and Gajic (1984) and Kokotovic and Cruz (1969). It is shown in Kokotovic and Cruz (1969) that an $O(\varepsilon^N)$ approximation of coefficients for the deterministic linear regulator problem implies the $O(\varepsilon^{2N})$ approximation of the corresponding criterion ($\Delta\sigma = O(\varepsilon^{2N})$). The same problem for the singularly perturbed linear stochastic regulator is studied in Khalil and Gajic (1984) which produces the relative error of $\Delta\sigma/\sigma = O(\varepsilon^N)$. In this paper we show that for the weakly coupled LQG problem an $O(\varepsilon^N)$ approximation of coefficients implies the absolute error of $\Delta\sigma = O(\varepsilon^N)$.

4. *Numerical example*

In order to demonstrate the numerical behavior of the near-optimum design of a weakly coupled LQG regulator, we present results for an LQG controller of a power system composed of two interconnected areas ($\dim A_{11} = 5 \times 5$, $\dim A_{22} = 4 \times 4$), (Geromel and Peres, 1985). The system model is given by:

$$A^{9\times 9} = \{a_{12} = 0.55,\ a_{23} = a_{67} = 1,\ a_{53} = a_{97} = -5.2,$$
$$a_{32} = a_{44} = a_{76} = a_{88} = -3.3,\ a_{33} = a_{77} = -0.05,$$
$$a_{34} = a_{78} = 6.0,\ a_{36} = a_{45} = a_{72} = a_{89} = 3.3,$$
$$a_{44} = a_{99} = -13.0,\ a_{16} = -0.55,\ \text{all other } a_{ij} = 0\}.$$
$$B^{9\times 2} = \{b_{51} = b_{92} = 13.0,\ \text{all other } b_{ij} = 0\}.$$
$$C^{4\times 9} = \{c_{11} = c_{24} = c_{48} = 1,\ c_{21} = c_{36} = 0.43,\ c_{31} = -1,$$
$$\text{all other } c_{ij} = 0\}.$$
$$D^{T}D^{9\times 9} = \{d_{11} = d_{33} = d_{77} = 1,\ d_{22} = d_{66} = 2,$$
$$d_{26} = d_{62} = -0.3,\ \text{all other } d_{ij} = 0\}$$
$$R = I_2.$$

It is assumed that $G = B$, and that white noise intensity matrices are given by

$$W_1 = 0.1,\quad W_2 = 0.1,\quad V_1 = I_2,\quad V_2 = I_2.$$

We can note relatively big elements in the cross coupling matrices A_{12}, A_{21} and C_{21}. The small parameter ε is built in the problem. The value for ε should be estimated from the problem's strongest coupled matrix—in this case matrix C. It seems from our experience that the formula

$$\varepsilon = \frac{\max(\|C_{12}\|, \|C_{21}\|)}{\max(\|C_{11}\|, \|C_{22}\|)} = \frac{1}{1.43} = 0.699 \qquad (4.1)$$

produces a quite good estimate for ε, where $\|\ \|$ is any suitable norm. In this example we have used the infinity norm.

It is important to note that there is no known method in the literature which produces an upper bound for the small parameter ε. This is true for the entire theory of small parameters (weak coupling, Gajic *et al.*, 1990; and singular perturbations, Kokotovic and Khalil, 1986). It happens that in this particular example, despite the relatively large value for the small parameter ε, the proposed method converges, since the radius of convergence of all algorithms used is less than 1 at each iteration (Gajic *et al.*, 1990). Simulation results are presented in the following table.

It is important to note that the approximate filters obtained with accuracy of $O(\varepsilon^2)$ and $O(\varepsilon^4)$ do not stabilize the plant-filter augmented system. Thus, in this example, if we wish to take advantage of the reduced-order suboptimal filters (on-line savings), we must obtain all coefficients (off-line calculations, Gajic *et al.*, 1990) with the accuracy of at least $O(\varepsilon^6)$. Table 1 verifies the result of Theorem 2, namely $\sigma - \sigma^{(k)} = O(\varepsilon^k)$, and supports the formula (4.1) for the estimate of the weak coupling parameter.

Simulation results are obtained by using the package L-A-S (West *et al.*, 1985) for the computer-aided control system design.

TABLE 1. APPROXIMATE VALUES FOR THE PERFORMANCE INDEX

k	$\sigma^{(k)}$	$\sigma^{(k)} - \sigma$	$(0.7)^k$
2	∞	∞	*
4	∞	∞	*
6	5.9415	0.9645	0.11765
10	5.1111	0.1341	0.02825
18	4.9788	0.0018	0.00163
26	4.9770	$<10^{-4}$	9.4×10^{-5}
Optimal	4.9770		

5. *Conclusion*

A new approach to simplifying LQG optimal design at steady-state for weakly coupled linear systems is developed. The new approach is conceptually simple and produces considerable savings in the size of on-line computations since in the approximate controls only reduced-order filters are used.

References

Chang, K. (1972). Singular perturbations of a general boundary value problem. *SIAM J. Math. Anal.*, **3**, 520–526.

Delacour, J. D., M. Darwish and J. Fantin (1978). Control strategies of large-scale power systems. *Int. J. Control*, **27**, 753–767.

Gajic, Z. (1986). Numerical fixed point solution for near optimum regulators of linear quadratic Gaussian control problems for singularly perturbed systems. *Int. J. Control*, **43**, 373–387.

Gajic, Z., Dj. Petkovski and X. Shen (1990). *Singularly Perturbed and Weakly Coupled Linear Control Systems—A Recursive Approach*, Springer, Berlin.

Gajic, Z. and X. Shen (1989). Decoupling transformation for weakly coupled linear systems. *Int. J. Control*, **50**, 1517–1523.

Geromel, J. and P. Peres (1985). Decentralized load-frequency control. *Proc. IEE*, **132**, 225–230.

Grodt, T. and Z. Gajic (1988). The recursive reduced-order numerical solution of the singularly perturbed matrix differential Riccati equation. *IEEE Trans. Aut. Control*, **AC-33**, 751–754.

Harkara, N., Dj. Petkovski and Z. Gajic (1989). The recursive algorithm for optimal output feedback control problem of linear weakly coupled systems. *Int. J. Control*, **50**, 1–11.

Ishimatsu, T., A. Mohri and M. Takata (1975). Optimization of weakly coupled systems by a two-level method. *Int. J. Control*, **22**, 877–882.

Khalil, H. and Z. Gajic (1984). Near-optimum regulators for stochastic linear singularly perturbed systems. *IEEE Trans. Aut. Control*, **AC-29**, 531–541.

Khalil, H. and P. Kokotovic (1978). Control strategies for decision makers using different models of the same system. *IEEE Trans. Aut. Control*, **AC-23**, 289–298.

Kokotovic, P. and J. Cruz Jr (1969). An approximation theorem for linear optimal regulators. *J. Math. Anal.*, **27**, 249–252.

Kokotovic, P. and H. Khalil (1986). *Singular Perturbations in Systems and Control*, IEEE Press, New York.

Kokotovic, P., W. Perkins, J. Cruz Jr and D'Ans (1969). ε-coupling for near-optimum design of large scale linear systems. *Proc. IEE.*, **116**, 889–992.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley-Interscience, New York.

Mahmoud, M. (1978). A quantitative comparison between two decentralized control approaches. *Int. J. Control*, **28**, 261–275.

Petkovski, Dj. and M. Rakic (1979). A series solution of feedback gains for output constrained regulators. *Int. J. Control*, **29**, 661–669.

Petrovic, B. and Z. Gajic (1988). Recursive solution of linear-quadratic Nash games for weakly interconnected systems. *J. Optimiz. Theory Applic.*, **56**, 463–477.

Sezer, M. and D. Siljak (1986). Nested ε-decomposition and clustering of complex systems. *Automatica*, **22**, 321–331.

Shen, X. (1990). Near-optimum reduced-order stochastic control of linear discrete and continuous systems with small parameters. Ph.D. dissertation, Rutgers University.

Washburn, H. D. and J. Mendel (1980). Multistage estimation of dynamical and weakly coupled systems in continuous-time linear systems. *IEEE Trans. Aut. Control,* **AC-25,** 71–76.

West, P., S. Bingulac and W. Perkins (1985). *L-A-S: A computer-aided control system design language.* In Jamshidi, M. and C. Herget (Eds), Computer Aided Control System Engineering. North-Holland, Amsterdam.

Zangwill, W. and C. Garcia (1981). *Pathways to Solutions, Fixed Points and Equilibria.* Prentice-Hall, Englewood Cliffs, NJ.