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Decoupling transformation for weakly coupled linear systems

ZORAN GAJIC† and XUEMIN SHEN†

The non-singular transformation that completely decouples a weakly coupled linear system under non-restrictive conditions is defined. The transformation matrices are obtained from two algebraic matrix equations. Algorithms that efficiently generate solutions to these equations are derived. The proposed transformation also completely decouples the corresponding Lyapunov matrix differential equation.

1. Introduction

The linear weakly coupled system represented by

$$\dot{x} = A_1 x + \varepsilon A_2 z + B_1 u_1 + \varepsilon B_2 u_2 \tag{1}$$

$$\dot{z} = \varepsilon A_3 x + A_4 z + \varepsilon B_3 u_1 + B_4 u_2 \tag{2}$$

where $x \in \mathbb{R}^{n_1}$, $z \in \mathbb{R}^{n_2}$, $u_i \in \mathbb{R}^{m_i}$, $i = 1, 2$, and ε is a small parameter has been studied in different set-ups by many researchers (Kokotovic *et al.* 1969, Delacour *et al.* 1978, Petkovski and Rakic 1979, Mahmoud 1978, Petrovic and Gajic 1988, Harkara *et al.* 1989, Gajic and Rayavarupu 1989, Sezar and Siljak 1986, Ishimatsu 1975, Washburn and Mendel 1980, Khalil and Kokotovic 1978). The main control equations of weakly coupled linear systems (Riccati type or Lyapunov type) have been studied from the power series expansion point of view by Kokotovic *et al.* (1969), Delacour *et al.* (1978), Petkovski and Rakic (1979), and Mahmoud (1978). A different recursive approach, based on the fixed point iterations, has been developed by Petrovic and Gajic (1988), Harkara *et al.* (1989), and Gajic and Rayavarupu (1989). In this work, we will introduce a new method that is the non-singular transformation that completely decouples linear weakly coupled systems (filters or estimators first of all). The main motivation for this transformation is the existence of the corresponding one for the other class of small parameter linear systems—singularly perturbed systems (Chang 1972, Kokotovic and Khalil 1986). In addition, the proposed transformation completely decouples the corresponding Lyapunov differential matrix equation.

2. Main result

Introducing the change of variables

$$x = \eta + \varepsilon L z \tag{3}$$

the original system (1) is transformed into

$$\dot{\eta} = A_{10}\eta + \varepsilon F_1(L) z + B_{10} u_1 + \varepsilon B_{20} u_2 \tag{4}$$

where

$$A_{10} = A_1 - \varepsilon^2 L A_3 \tag{5}$$

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$$B_{10} = B_1 - \varepsilon^2 L B_3, \quad B_{20} = B_2 - L B_4 \tag{6}$$

$$F_1(L) = A_1 L - L A_4 + A_2 - \varepsilon^2 L A_3 L \tag{7}$$

Assuming that a matrix L can be chosen such that $F_1(L) = 0$, (4) will represent a completely independent (decoupled) subsystem

$$\dot{\eta} = A_{10}\eta + B_{10}u_1 + \varepsilon B_{20}u_2 \tag{8}$$

As a matter of fact, (2) and (8) form a tridiagonal system—after elimination of x from (2) by using (3).

Introducing a second change of variables as

$$\zeta = z + \varepsilon H \eta \tag{9}$$

(2) becomes

$$\dot{\zeta} = A_{40}\zeta + \varepsilon F_2(H)\eta + \varepsilon B_{30}u_1 + B_{40}u_2 \tag{10}$$

where

$$A_{40} = A_4 + \varepsilon^2 A_3 L \tag{11}$$

$$B_{30} = B_3 + H B_{10}, \quad B_{40} = B_4 + \varepsilon^2 H B_{20} \tag{12}$$

$$F_2(H) = H A_{10} - A_{40} H + A_3 \tag{13}$$

In addition, if a matrix H can be chosen such that $F_2(H) = 0$, then we have

$$\dot{\zeta} = A_{40}\zeta + \varepsilon B_{30}u_1 + B_{40}u_2 \tag{14}$$

so that (8) and (14) represent two completely decoupled linear subsystems. Notice that the weakly coupled structure of the control inputs in (1) and (2) is preserved in the new coordinates, that is in (8) and (14). This means that the proposed transformation is applicable to the feedback structure of (1) and (2) also. Thus, applying the non-singular transformation

$$\begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} I & -\varepsilon L \\ \varepsilon H & I - \varepsilon^2 H L \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = T \begin{bmatrix} x \\ z \end{bmatrix} \tag{15}$$

where

$$T^{-1} = \begin{vmatrix} I - \varepsilon^2 L H & \varepsilon L \\ -\varepsilon H & I \end{vmatrix} \tag{16}$$

the linear weakly coupled system (1) and (2) is completely decoupled and uniquely determined by its subsystems (8) and (14).

Obviously, the transformation T is uniquely obtained if unique solutions of the following two algebraic equations exist:

$$A_1 L - L A_4 + A_2 - \varepsilon^2 L A_3 L = 0 \tag{17}$$

$$H(A_1 - \varepsilon^2 L A_3) - (A_4 + \varepsilon^2 A_3 L) H + A_3 = 0 \tag{18}$$

It is important to notice that at $\varepsilon = 0$ we have

$$A_1 L^{(0)} - L^{(0)} A_4 + A_2 = 0 \tag{19}$$

$$H^{(0)} A_1 - A_4 H^{(0)} + A_3 = 0 \tag{20}$$

so that

$$L = L^{(0)} + 0(\varepsilon^2) \tag{21}$$

$$H = H^{(0)} + 0(\varepsilon^2) \tag{22}$$

Equations (19) and (20) are Sylvester equations and their unique solution exist if matrices A_1 and A_4 have no eigenvalues in common (Lancaster and Tismenetsky 1985, p. 414). Then by the implicit function theorem (Ortega and Rheinboldt 1970, p. 128) for a sufficiently small $\varepsilon \in (0, \varepsilon_1]$ there exists a unique solution of a weakly non-linear algebraic equation (17). Under the assumption that A_1 and A_4 have no eigenvalues in common and by the fact that the eigenvalues are continuous functions of the matrix elements (Kato 1980), there exists ε_2 small enough such that for any $\varepsilon \in (0, \varepsilon_2]$ matrices A_{10} and A_{40} will not have eigenvalues in common and thus, the unique solution of (18) will exist.

In summary, we have established the following theorem.

Theorem 1

Under the assumption that the matrices A_1 and A_4 have no eigenvalues in common there exists a small parameter $\varepsilon \in (0, \min(\varepsilon_1, \varepsilon_2)]$ such that the unique solutions of (17) and (18) exist.

Trajectories of the transformed (decoupled) system are $O(\varepsilon)$ close to the trajectories of the original system. If the coupling parameter ε is extremely small, or if in the design procedure the accuracy of $O(\varepsilon)$ is sufficient, there is no need for the decomposition. However, if $O(\varepsilon)$ is not very small, or if the high accuracy is required, then one needs methods that will produce any desired accuracy, that is the accuracy of $O(\varepsilon^k)$ where $k = 2, 3, 4, \dots$. Thus, the method proposed in this work is very useful for the intermediate values of ε and for the systems with the high accuracy requirements. In addition, the importance of the proposed transformation is in the design of linear filters and observers—dynamical systems built by the designer. Apparently, it is much easier and less expensive to build two dynamical systems of order n_1 and n_2, than one dynamical system of order $n_1 + n_2$.

Note that the transformations (3) and (9) can be used independently to put the system in either lower or upper triangular form. For some applications, this might be sufficient.

3. Numerical methods for L equation

Numerical solutions for L and H can be obtained by using the fixed point type recursive algorithm similar to those developed by Petrovic and Gajic (1988), Harkara *et al.* (1989), Gajic and Rayavarupu (1989), and Gajic (1986). In the case of (17) and (18) the corresponding algorithm is given by

$$A_1 L^{(i+1)} - L^{(i+1)} A_4 + A_2 - \varepsilon^2 L^{(i)} A_3 L^{(i)} = 0 \tag{23}$$

with $i = 0, 1, 2, \dots, N-1$, and $L^{(0)}$ obtained from (19)

$$H^{(N)} A_{10}^{(N)} - A_{40}^{(N)} H^{(N)} + A_3 = 0 \tag{24}$$

where

$$A_{10}^{(N)} = A_1 - \varepsilon^2 L^{(N)} A_3, \quad A_{40}^{(N)} = A_4 + \varepsilon^2 A_3 L^{(N)}$$

Using the results of the references given above, it can be shown that

$$L = L^{(N)} + O(\varepsilon^{2N}) \tag{25}$$

and

$$H = H^{(N)} + O(\varepsilon^{2N}) \tag{26}$$

hence, the algorithm (23) converges with the rate of convergence of $O(\varepsilon^2)$.

Example 1

In order to demonstrate the efficiency of the proposed algorithm (23), we have run a sixth-order example. Matrices A_i, $i = 1, 2, 3, 4$ are chosen randomly (standard deviation = 1 and mean value = 0 for A_1, A_2, and A_3; standard deviation = 2 and mean value = 0 for A_4)

$$A_1 = \begin{bmatrix} -1.720 & -0.999 & -0.592 \\ -1.434 & 0.779 & 0.856 \\ -0.729 & 0.105 & 0.867 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.614 & -1.429 & 0.516 \\ 0.225 & 1.928 & 0.310 \\ -0.332 & 0.067 & 0.329 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.398 & 1.039 & 0.557 \\ 1.298 & 1.349 & -0.891 \\ -0.472 & -0.610 & -0.873 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2.956 & 1.219 & 2.269 \\ -0.038 & -2.240 & 2.296 \\ -0.873 & -2.020 & 2.344 \end{bmatrix}$$

The simulation results for different values of the coupling parameters ε are given in Table 1.

ε	Number of required iterations such that $\lVert L^{(i+1)} - L^{(i)} \rVert_\infty < 10^{-10}$
0.8	*
0.7	28
0.6	17
0.5	12
0.3	9
0.1	5
0.05	3
0.01	2

*denotes non-convergence

Table 1. Dependence of number of iterations on ε.

The results of Table 1 strongly support the necessity for the existence of the recursive scheme for the solution of (17), since unless ε is very small, the zeroth- and first-order approximations are far from the optimal solution.

In Table 2, we show the propagation of the error per iteration when $\varepsilon = 0.1$. We notice that the rate of convergence of the proposed algorithm (23) is $O(\varepsilon^2) = O(10^{-2})$.

The algorithm (23) is based on the fixed point iterations, and it will converge so long as the small parameter ε is small enough such that the radius of convergence $\rho(\varepsilon) < 1$ at each iteration.

An alternative way of solving (17) is by using the Newton method where solution of (19) plays the role of the initial condition. The Newton method for the similar type of algebraic equation has been presented by Grodt and Gajic (1988). The Newton

$\varepsilon = 0{\cdot}1$	
i	$\lVert L^{(i+1)} - L^{(i)} \rVert_\infty$
0	$4{\cdot}1290 \times 10^{-2}$
1	$7{\cdot}4645 \times 10^{-4}$
2	$1{\cdot}6401 \times 10^{-5}$
3	$2{\cdot}1149 \times 10^{-7}$
4	$2{\cdot}0989 \times 10^{-9}$

Table 2. Propagation of error per iteration.

algorithm for (17) can be constructed by setting $L^{(i+1)} = L^{(i)} + \Delta L^{(i)}$ and neglecting $O((\Delta L)^2)$ terms. This will produce a Sylvester-type equation of the form

$$D_1^{(i)} L^{(i+1)} + L^{(i+1)} D_2^{(i)} = Q^{(i)}, \quad i = 0, 1, 2, \ldots \tag{27}$$

where

$$D_1^{(i)} = (A_1 - \varepsilon^2 L^{(i)} A_3)$$

$$D_2^{(i)} = -(A_4 + \varepsilon^2 A_3 L^{(i)})$$

$$Q^{(i)} = -(A_2 + \varepsilon^2 L^{(i)} A_3 L^{(i)})$$

with the initial condition $L^{(0)}$ obtained from (19).

The Newton method is demonstrated by solving the same example. For the different values of ε the results are presented in Table 3.

ε	Number of iterations such that $\lVert L^{(i+1)} - L^{(i)} \rVert_\infty < 10^{-10}$
0·8	5
0·7	5
0·6	4
0·5	4
0·3	3
0·1	2
0·05	2
0·01	1

Table 3. Solutions by the Newton method.

It can be seen, that for this particular example, the Newton method converges much faster than the fixed point iteration algorithm. It is a well-known fact that the Newton method converges quadratically in the neighbourhood of the sought solution and that its main problem lies in the choice of the initial guess. For the algebraic equation (17) the initial guess is easily obtained with the accuracy of $O(\varepsilon^2)$, and the Newton method, if it converges, will produce a sequence $O(\varepsilon^4)$, $O(\varepsilon^8)$, $O(\varepsilon^{16})$, close to the exact solution. However, in some cases the Newton method does not converge at all (bad initial guess) and one needs to have some other efficient techniques available.

The fixed point method presented earlier in this work is one of them, since its rate of convergence of $O(\varepsilon^2)$ is remarkable.

The simulation results are obtained by using a software package L–A–S (West *et al.* 1985) for computer-aided control system design.

In the next section we will show that the introduced transformation T completely decouples the Lyapunov matrix differential equation as well.

4. Decomposition of the differential matrix Lyapunov equation

Consider the Lyapunov matrix differential equation of weakly coupled systems

$$\dot{P} = A^{\mathrm{T}} P + PA + Q, \quad Q = Q^{\mathrm{T}}, \quad P(t_0) = P_0 \tag{27}$$

where the given matrices A and Q are partitioned as

$$A = \begin{bmatrix} A_1 & \varepsilon A_2 \\ \varepsilon A_3 & A_4 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & \varepsilon Q_2 \\ \varepsilon Q_2^{\mathrm{T}} & Q_3 \end{bmatrix}$$

Due to the assumed structure for A and Q, the matrix P is properly scaled as (Kokotovic *et al.* 1969)

$$P = \begin{bmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^{\mathrm{T}} & P_3 \end{bmatrix} \tag{28}$$

Multiplying (27) from the left by $T^{-\mathrm{T}}$ and from the right-hand side by T^{-1} we get

$$T^{-\mathrm{T}} \dot{P} T^{-1} = T^{-\mathrm{T}} A^{\mathrm{T}} P T^{-1} + T^{-\mathrm{T}} P A T^{-1} + T^{-\mathrm{T}} Q T^{-1} \tag{29}$$

which can be written as

$$\dot{K} = a^{\mathrm{T}} K + K a + q, \quad K(t_0) = K_0 \tag{30}$$

where

$$a = TAT^{-1} = \begin{bmatrix} A_{10} & 0 \\ 0 & A_{40} \end{bmatrix} \tag{31}$$

$$q = T^{-\mathrm{T}} Q T^{-1} = \begin{bmatrix} q_1 & \varepsilon q_2 \\ \varepsilon q_2^{\mathrm{T}} & q_3 \end{bmatrix} \tag{32}$$

with

$$K = T^{-\mathrm{T}} P T^{-1} = \begin{bmatrix} K_1 & \varepsilon K_2 \\ \varepsilon K_2^{\mathrm{T}} & K_3 \end{bmatrix}, \quad K(t_0) = T^{-\mathrm{T}} P_0 T^{-1} \tag{33}$$

Partitioning (30) we note a completely decoupled form among elements of K, thus

$$\dot{K}_1 = K_1 A_{10} + A_{10}^{\mathrm{T}} K_1 + q_1 \tag{34 a}$$

$$\dot{K}_2 = K_2 A_{40} + A_{10}^{\mathrm{T}} K_2 + q_2 \tag{34 b}$$

$$\dot{K}_3 = K_3 A_{40} + A_{40}^{\mathrm{T}} K_3 + q_3 \tag{34 c}$$

Having obtained the K_i from (34), we can get the solution of the Lyapunov differential equation in the original coordinates as

$$P = T^{\mathrm{T}} K T \tag{35}$$

5. Conclusion

The powerful transformation that completely decouples weakly coupled linear systems is introduced. Besides its theoretical importance (for example in the stability study and in the study of the variance of linear systems driven by white noise), it can be used in the practical implementation of linear filters and observers.

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