Dynamics of Continuous, Discrete and Impulsive Systems
Series **B:** *Applications & Algorithms* **8 (2001)** 359-373

STABILITY ANALYSIS OF A CLASS OF HYBRID DYNAMIC SYSTEMS

Xinzhi Liu[1], Xuemin Shen[2] and Y. Zhang[1]
[1]Department of Applied Mathematics
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1
[2]Department of Electrical and Computer Engineering
University of Waterloo
Waterloo, Ontario, Canada N2L 3G1

Abstract. This paper studies the stability issue of a class of nonlinear hybrid dynamic systems, namely, dynamic systems with jump parameters. The hybrid system model is described first and compared with the conventional one. Sufficient conditions are established for the stability and asymptotic stability of the nonlinear hybrid systems. Extension to hybrid systems with time delay are also investigated. Stability criteria are obtained for both linear and nonlinear cases. Examples are given to illustrate the main results.
AMS subject classifications: 34D20,34K20.

1 Introduction

Fault prone dynamic systems may experience abrupt changes in their structures and parameters, caused by phenomena such as component failure or repairs, changing subsystem interconnections, and abrupt environmental disturbances. Such systems can be modeled as operation in different forms [1], where each form corresponds to some combination of these events. A mathematical model describing such phenomena is given by

$$x'(t) = A(r(t))x(t), \tag{1.1}$$

where $x'(t)$ is the derivative of $x(t)$, $r(t) \in S = \{1, 2, \cdots, N\}$. $r(t)$ may jump from i to j following some rules or randomly from time to time. For each $r(t) = i$, $t \in [t_i, t_i + \sigma_i)$, $A(r(t)) = A(i)$ has different form. If $r(t) = i$ in the internal $[t_i, t_i + \sigma_i)$, system (1.1) is be in the ith state in that interval. System (1.1) belongs to the category of hybird systems, since it combines a part of the state that takes values continuously ($x \in R^n$) and another part of the state that takes discrete values ($r(t) \in \{1, 2, \cdots, N\}$). In general, it is not known exactly when the system jumps from ith$(r(t) = i)$ state to jth$(r(t) = j)$ state, but the probability or relating coefficient that the system jumps from ith state to jth state is certain number α_{ij}, $\alpha_{ij} \geq 0$, $i, j = 1, 2, \cdots, N$. Such systems are often called systems with jump parameters. For simplicity, we shall call them jump systems. Clearly, jump systems are different from conventional dynamic systems given by

$$x'(t) = A(t)x(t) \tag{1.2}$$

because a jump system may change the state from ith to jth abruptly and undeterminately. For a given state, say ith state, the system is determinate. If the jump system remains in the ith state in interval $[t_{i_1}, t_{i_2})$, then the system can be treated as a determinate system in $[t_{i_1}, t_{i_2})$. Jump systems have been studied extensively in the field of control technology, signal tracking, etc. [2]–[11], since they exhibit mixture of continuous dynamics and discrete events, and provide a general framework for mathematical modeling of many real world phenomena. Most studies focus on the linear jump systems only due to the model complexity.

In this paper, we shall study the stability of nonlinear jump systems and jump systems with time delays. The remainder of this paper is organized as follows. Section 2 gives notations and stability definitions. In section 3, sufficient conditions are established for stability and asymptotic stability of nonlinear jump systems. These conditions are extended to jump systems with time delays in section 4. Several stability criteria are obtained for both the linear and nonlinear jump systems. Examples are presented in sections 3 and 4 to illustrate the main results. Conclusions are given in section 5.

2 Preliminaries

Consider the nonlinear jump system

$$x'(t) = A(r(t))x(t) + f(t, x(t), r(t)) \tag{2.3}$$

where $A(r(t))$ is an $n \times n$ matrix for any $r(t) \in S = \{1, 2, \cdots, N\}$. $f(t, x, r(t))$ is continuous with respect to t and x for every fixed state $r(t) = i$ and $f(t, 0, r(t)) \equiv 0$.

Let $R_+ = [0, \infty)$, A^T be the transpose of A, $x(t, t_0, x_0, r(t))$ be the solution of system (2.3) with initial value $x(t_0, t_0, x_0, r(t_0)) = x_0$, and $||x||$ be the Euclidean norm, i.e., $||x|| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. A matrix P is called positive definite if $P^T = P$ and all of its eigenvalues are positive. A positive definite matrix P is denoted by $P > 0$. A function $W(x)$ is called positive definite if $W \in C(R^n, R_+)$, $W(0) = 0$ and $W(x) > 0$ when $x \neq 0$. A function $V(t, x)$ is called positive definite if $V \in C(R \times R^n, R_+)$, and there exists a positive definite function $W(x)$ such that $V(t, x) \geq W(x)$.

Definition 2.1 *System (2.3) is called stable if for given t_0 and any $\epsilon > 0$ there exists a $\delta > 0$ such that $||x_0|| < \delta$ implies that $||x(t, t_0, x_0, r(t))|| < \epsilon, t \geq t_0$. Otherwise, the system is called unstable.*

Definition 2.2 *System (2.3) is called asymptotically stable if it is stable and there exists $\sigma > 0$ such that $||x_0|| < \sigma$ implies that solutions $x(t, t_0, x_0, r(t))$ of (2.3) satisfy*

$$\lim_{t \to \infty} x(t, t_0, x_0, r(t)) = 0.$$

The following example illustrates the difference between jump systems and determinate systems.

Example 2.1 *Consider the systems*

$$x'(t) = Ax(t); \tag{2.4}$$

and

$$y'(t) = By(t). \tag{2.5}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 1 \\ -9 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}.$$

The general solutions of (2.4) *and* (2.5) *are given by*

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{-t} \begin{pmatrix} c_1 cos3t + c_2 sin3t \\ -3c_1 sin3t + 3c_2 cos3t \end{pmatrix}$$

and

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{-t} \begin{pmatrix} 3c_1 sin3t - 3c_2 cos3t \\ c_1 cos3t + c_2 sin3t \end{pmatrix}$$

respectively. Furthermore, the solution of (2.4) *with the initial condition* $x_1(t_0) = x_0,\ x_2(t_0) = 0$ *and the solution of* (2.5) *with the initial condition* $y_1(t_0) = 0, y_2(t_0) = y_0$ *are*

$$\begin{cases} x_1(t) = x_0 e^{-(t-t_0)} cos3(t - t_0) \\ x_2(t) = -3x_0 e^{-(t-t_0)} sin3(t - t_0) \end{cases}, \quad \begin{cases} y_1(t) = 3y_0 e^{-(t-t_0)} sin3(t - t_0) \\ y_2(t) = y_0 e^{-(t-t_0)} cos3(t - t_0) \end{cases} \tag{2.6}$$

respectively. It can be seen that both systems (2.4) *and* (2.5) *are asymptotically stable. Let us consider the jump system given by*

$$z'(t) = A(r(t))z(t) \tag{2.7}$$

where $r(t) \in \{1, 2\}$, $z = (z_1, z_2)^T$ *and*

$$A(1) = A = \begin{pmatrix} -1 & 1 \\ -9 & -1 \end{pmatrix}, \quad A(2) = B = \begin{pmatrix} -1 & 9 \\ -1 & -1 \end{pmatrix}.$$

Let $t_0 = 0$, $r(t) = 1$ *for* $t \in [2m\pi/6, (2m + 1)\pi/6)$ *and* $r(t) = 2$ *for* $t \in [(2m+1)\pi/6, (2m+2)\pi/6)$, *where* m *is an integer. Then the solution* $z(t) =$

$z(t, 0, z^0, r(t))$ *of system* (2.7) *with the initial condition* $z^T(0) = (z_0, 0)^T$ *is*

$$\begin{cases} z_1(t) = (-1)^m 3^{2m} z_0 e^{-t} cos3(t - 2m\pi/6) \\ z_2(t) = (-1)^{m+1} 3^{2m+1} z_0 e^{-t} sin3(t - 2m\pi/6), \\ \qquad t \in [2m\pi/6, (2m+1)\pi/6), \end{cases}$$

$$\begin{cases} z_1(t) = (-1)^{m+1} 3^{2(m+1)} z_0 e^{-t} sin3(t - (2m+1))\pi/6) \\ z_2(t) = (-1)^{m+1} 3^{2m+1} z_0 e^{-t} cos3(t - (2m+1)\pi/6), \\ \qquad t \in [(2m+1)\pi/6, 2(m+1)\pi/6), \end{cases}$$

in view of (2.6). *It is obvious that, by the definition, system* (2.7) *is unstable.*

This example shows that the stability of the jump system in each ith state for all $i = 1, \ldots, N$, can not guarantee the stability of the jump system. It also illustrates that jump system is different from determinate system although it can be considered as determinate system in any fixed interval when the jump system has no state transition.

3 Stability Criteria

Stability criteria for system (2.3) are established in this section.

Theorem 3.1 *Assume that*

(i) there exist $n \times n$ *symmetric matrices* $Q_i, i = 1, 2, \ldots, N$, *such that the solutions* P_i *of the systems*

$$A^T(i)P_i + P_iA(i) + Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j = 0$$

are positive definite;

(ii) $x^TP_ix \le y^TP_iy$ *implies* $||x|| \le ||y||$ *for all* $x, y \in R^n$ *and* P_i, $i = 1, 2, \cdots, N$;

(iii) there exist $m_i \in C(R, R), i = 1, \cdots, N$, *and* $-K \in C([t_0, \infty), R_+)$ *such that*
$2f(t, x, i)^TP_ix \le m_i(t)x^Tx$
and
$-x^T[Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j - m_i(t)I]x \le K(t)x^Tx$,

where I *is an identity matrix.*

Then system (2.3) *is stable if* $\int_{t_0}^{\infty} K(s)ds > -\infty$ *and it is asymptotically stable if* $\int_{t_0}^{\infty} K(s)ds = -\infty$.

Proof: Since $P_i > 0$, then there exists $\lambda_1, \lambda_2 > 0$ such that $\lambda_2 x^T x \leq x^T P_i x \leq \lambda_1 x^T x$ for $i = 1, 2, \cdots, N$.

Let

$$V(x, r(t)) = V_i = x_i^T P_i x_i, \quad for \quad r(t) = i.$$

The derivative of V_i along solutions of system (2.3) is given by

$$\begin{aligned}
V_i' &= (x^T)' P_i x + x^T P_i x' \\
&= (x^T A^T(i) + f^T(t, x, i)) P_i x + x^T P_i (A(i)x + f(t, x, i)) \\
&= x^T [A^T(i) P_i + P_i A(i)] x + 2 f^T(t, x, i) P_i x \\
&= -x^T [Q_i + \textstyle\sum_{j=1}^{N} \alpha_{ij} P_j] x + 2 f^T(t, x, i) P_i x \\
&\leq -x^T [Q_i + \textstyle\sum_{j=1}^{N} \alpha_{ij} P_j] x + m_i(t) x^T x \\
&= x^T [-Q_i - \textstyle\sum_{j=1}^{N} \alpha_{ij} P_j + m_i(t) I] x \\
&\leq K(t) x^T x \leq \lambda_1^{-1} K(t) V_i.
\end{aligned}$$

Thus for any solution $x(t) = x(t, t_0, x_0, r(t))$

$$\begin{aligned}
x^T(t) P_i x(t) &= V_i \leq V(t_i, x(t_i), i) e^{\lambda_1^{-1} \int_{t_i}^{t} K(s)ds} \\
&= [x^T(t_i) e^{(\lambda_1^{-1} \int_{t_i}^{t} K(s)ds)/2}] P_i [x(t_i) e^{(\lambda_1^{-1} \int_{t_i}^{t} K(s)ds)/2}] \qquad (3.8) \\
& \qquad t \in [t_i, t_i + \sigma),
\end{aligned}$$

where t_i is a jump point of system (2.3) such that (2.3) stays in the ith state in interval $[t_i, t_i + \sigma)$. From (3.8) and in view of the condition (ii), we have

$$||x(t)|| \leq ||x(t_i)|| e^{(\lambda_1^{-1} \int_{t_i}^{t} K(s)ds)/2}, \quad [t_i, t_i + \sigma). \qquad (3.9)$$

Let $t_i < t_j < t_k$ be successive jump points, i.e., system (2.3) is in the ith state in interval $[t_i, t_j)$, and in the jth state in interval $[t_j, t_k)$. By (3.9), we have

$$\begin{aligned}
||x(t)|| &\leq ||x(t_j)|| e^{(\lambda_1^{-1} \int_{t_j}^{t} K(s)ds)/2} \\
&\leq ||x(t_i)|| e^{(\lambda_1^{-1} \int_{t_i}^{t} K(s)ds)/2}, \quad t \in [t_i, t_k).
\end{aligned}$$

By induction, we have

$$||x(t)|| \leq ||x_0||e^{(\lambda_1^{-1}\int_{t_0}^t K(s)ds)/2}, \quad t \in [t_0, \infty),$$

which yields the conclusions of the theorem.

The following result is a generalization of Theorem 3.1.

Theorem 3.2 *Assume that*

(i) $v_i \in C[R \times R^n, R_+]$ *and* $k \in C[R_+, R_+]$ *such that*

$$\begin{aligned} v'_{i(2.3)}(t, x, r(t)) &= \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial x}[A(r(t))x + f(t, x, r(t))] \\ &\leq -k(t)v_i(t, x, r(t)), \quad t \in I_i; \end{aligned} \tag{3.10}$$

(ii) if for $t_1, t_2 \in R_+, r(t_1) = r(t_2)$ *and* $v(t_1, x, r(t_1)) \leq v(t_2, y, r(t_2))$*, then* $||x|| \leq ||y||$.

Then system (2.3) *is stable if* $\int_{t_0}^{\infty} k(s)ds \leq \infty$ *and it is asymptotically stable if* $v_i(t, x, r(t))$ *is positive definite and* $\int_{t_0}^{\infty} k(s)ds = \infty$.

Proof: Let $x(t) = x(t, t_0, x_0)$ be any solution of system (2.3). We claim that for any $t \geq t_0$,

$$||x(t)|| \leq ||x(t_0)||. \tag{3.11}$$

In fact, for any two neighboring states of system (2.3), say $r(t) = i, t \in [t_{i_1}, t_{i_2})$ and $r(t) = j, t \in [t_{i_1}, t_{j_2}) = [t_{j_1}, t_{j_2})$, using condition (3.10) we have

$$\begin{aligned} v_i(t, x(t), r(t)) &= v_i(t, x(t), i) \\ &\leq v_i(t_{i_1}, x(t_{i_1}), i)e^{-\int_{t_{i_1}}^t k(s)ds}, \quad t \in [t_{i_1}, t_{i_2}); \\ v_j(t, x(t), r(t)) &= v_j(t, x(t), j) \\ &\leq v_j(t_{i_2}, x(t_{i_2}), j)e^{-\int_{t_{i_2}}^t k(s)ds} \\ &= v_j(t_{j_1}, x(t_{j_1}), j)e^{-\int_{t_{j_1}}^t k(s)ds}, \quad t \in [t_{j_1}, t_{j_2}), \end{aligned}$$

and hence

$$\begin{aligned} v_i(t, x(t), i) &\leq v_i(t_{i_1}, x(t_{i_1}), i), & t \in [t_{i_1}, t_{i_2}); \\ v_j(t, x(t), j) &\leq v_j(t_{i_2}, x(t_{i_2}), j) = v_j(t_{j_1}, x(t_{j_1}), j), & t \in [t_{j_1}, t_{j_2}). \end{aligned}$$

By condition (ii), we have

$$\begin{aligned} ||x(t)|| &\leq ||x(t_{i_1})|| && t \in [t_{i_1}, t_{i_2}); \\ ||x(t)|| &\leq ||x(t_{j_1})|| = ||x(t_{i_2})|| \\ &\leq ||x(t_{i_1})|| && t \in [t_{j_1}, t_{j_2}). \end{aligned}$$

Then, by induction, we have (3.11) which implies that the system (2.3) is stable. Thus for any $\sigma > 0$, $t_0 \in R_+$, there exists an $\rho = \rho(t_0, \sigma) > 0$ such that $||x_0|| < \sigma$ implies $||x(t)|| < \rho, t \geq t_0$, where $x(t) = x(t, t_0, x_0)$ is any solution of (2.3).

We show next that

$$lim_{t\to\infty}||x(t, t_0, r(t_0))|| = 0, \tag{3.12}$$

if $v_i(t, x(t), r(t))$ is positive definite and $\int_{t_0}^{\infty} k(s)ds = \infty$.

Let $t_0 \in I_l$, then

$$v_i(t, x(t), r(t)) \leq v_l(t_0, x(t_0), r(t_0))e^{-\int_{t_0}^{t} k(s)ds}, \quad t \in I_i \tag{3.13}$$

which implies

$$lim_{t\to\infty}x(t) = 0$$

since v is positive definite and $\int_{t_0}^{\infty} k(s)ds = \infty$, i.e., (3.12) is true. Thus system (2.3) is asymptotically stable. The proof is complete.

Example 3.1 *Consider the nonlinear jump system*

$$x'(t) = A(r(t))x(t) + f(t, x(t), r(t)) \tag{3.14}$$

where $x(t) \in R^2$, $r(t) = 1, 2$, *and,*

$$A(r(t))x(t) + f(t, x(t), r(t)) = \begin{pmatrix} -4 & 1 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} sinx_2(t) \\ cosx_1(t) \end{pmatrix},$$

for $r(t) = 1$;

$$A(r(t))x(t) + f(t, x(t), r(t)) = \begin{pmatrix} -4 & 5 \\ -1 & -6 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} x_1^3(t) \\ ln(1 + |x_2(t)|) \end{pmatrix},$$

for $r(t) = 2$.

By taking $v_i(x(t), r(t)) = [x_1^2(t) + x_2^2(t)]/2$ *for* $i = 1, 2$, *it is clearly seen that* $v_i(x(t), i) \leq v_i(y(t), i)$ *implies* $||x(t)|| \leq ||y(t)||$ *for all* $x(t), y(t) \in R^2$ *and* $i = 1, 2$.

The derivatives of v_i *along system* (3.14) *are*

$$\begin{aligned} v_1' &= x_1x_1' + x_2x_2' \\ &= x_1(-4x_1 + x_2 + sinx_2) + x_2(3x_1 - 3x_2 + cosx_1) \\ &= -4x_1^2 + 4x_1x_2 - 3x_2^2 + x_1sinx_2 + x_2cosx_1 \\ &\leq -4x_1^2 + 6|x_1x_2| - 3x_2^2 \\ &\leq -0.2v_1, \qquad forr(t) = 1, \end{aligned}$$

$$\begin{aligned} v_2' &= x_1x_1' + x_2x_2' \\ &= x_1(-4x_1 + 5x_2 - x_1^3) + x_2(-x_1 - 6x_2 + ln(1 + |x_2|)) \\ &\leq -4x_1^2 + 4x_1x_2 - 6x_2^2 - x_1^4 + x_2^2 \\ &\leq -4x_1^2 + 4|x_1x_2| - 5x_2^2 \\ &\leq -0.2v_2, \qquad forr(t) = 2, \end{aligned}$$

which satisfy all conditions of Theorem 3.2. Therefore, the jump system (3.14) *is asymptotically stable.*

4 Jump Systems with Time Delay

In this section, we shall extend our study to jump systems with time delay. Consider the following jump system

$$x'(t) = A(r(t))x(t) + B(r(t))x(t - \tau(t)) \tag{4.15}$$

where $A(r(t)), B(r(t))$ are $n \times n$ matrices for any $r(t) \in S = \{1, 2, \cdots, N\}$. $0 \leq \tau(t) \leq \tau, \tau = constant$. Let $C_t = \{x : \; x \in C([t-\tau, t], R^n)\}$. If $x_t \in C_t$, define $|||x_t||| = sup_{t-\tau \leq s \leq t}||x(t)||$.

The stability definitions for system (4.15) are similar to Definition 2.1 and Definition 2.2 except that $||x(t_0)|| < \delta$ and $||x(t, t_0, x_0, r(t))|| < \epsilon$ are replaced by $|||x_{t_0}||| < \delta$ and $||x(t, t_0, x_{t_0}, r(t))|| < \epsilon$ respectively, where $x_{t_0} \in C_{t_0}$ and $x(t, t_0, x_{t_0}, r(t))$ is the solution of system (4.15) with initial value $x_{t_0} = \phi(\theta), \; \theta \in [-\tau, \; 0]$.

Lemma 4.1 *Let* $v \in C^1[t_0, b], \; t_0 < b \leq \infty$ *such that*

$$v'(t) \leq -\alpha(t)v(t) + \beta(t)sup_{t-\tau \leq s \leq t}v(s), \quad t \in [t_0, b),$$

where $\alpha(t) \geq \beta(t) \geq 0$ *are continuous. If for any* $\epsilon > 0$,

$$S = \{t \; |v(t) = |||v_t||| \leq |||v_{t_0}||| + \epsilon, \quad t \in [t_0, T]\}.$$

Then, if S is not empty,

$$S = \cup[\alpha_i, \beta_i], \quad \alpha_{i+1} \geq \beta_i, \quad i = 1, 2, \cdots.$$

Proof Suppose that S is not empty. Let $\alpha_1 = infS$, $\beta_1 = inf\{t|t \geq \alpha_1, v'(t) < 0\}$. Then $v'(t) \geq 0, t \in [\alpha_1, \beta_1]$ and so $[\alpha_1, \beta_1] \subset S$ since $v(\alpha_1) = sup_{\alpha_1-\tau\leq s\leq\alpha_1} v(s)$, and $v(t)$ is an increasing function for $t \in [\alpha_1, \beta_1]$.

Let $\alpha_2 = inf\{t|t > \beta_1, t \in S\}$, $\beta_2 = inf\{t|t \geq \alpha_2, v'(t) < 0\}$. Then $v'(t) \geq 0$, $t \in [\alpha_2, \beta_2]$ and so $[\alpha_2, \beta_2] \in S$.

If $\beta_k < b$ and there exists $t \in S - \cup_{i=1}^{k}[\alpha_i, \beta_i]$, then let $\alpha_{k+1} = inf\{t|t > \beta_k, t \in S\}$, $\beta_{k+1} = inf\{t|t \geq \alpha_{k+1}, v'(t) < 0\}$. Then $v'(t) \geq 0$, $t \in [\alpha_{k+1}, \beta_{k+1}]$ and so $[\alpha_{k+1}, \beta_{k+1}] \in S$.

By induction, we have

$$S = \cup[\alpha_i, \beta_i].$$

The proof is completed.

Lemma 4.2 *Let $v(t)$ be a nonnegative, continuously differentiable function defined on $[t_0, b)$ such that*

$$v'(t) \leq -\alpha(t)v(t) + \beta(t) sup_{t-\tau\leq s\leq t} v(s),$$

for $t \in [t_0, b)$, where $\alpha(t) \geq \beta(t) \geq 0$ are continuous. Then

$$v(t) \leq sup_{t_0-\tau\leq s\leq t_0} v(s) = |||v_{t_0}|||.$$

Proof For any $\epsilon > 0$, we claim that

$$v(t) < |||v_{t_0}||| + \epsilon. \tag{4.16}$$

By continuity of $v(t)$ there exists a $\delta > 0$ such that $|||v(t)||| < |||v_{t_0}||| + \epsilon$, $t \in [t_0, t_0 + \delta]$. If (4.16) is not true, then there exists a $T > t_0$ such that $v(t) < v(T), t \in [t_0 - \tau, T)$ and $v(T) = |||v_{t_0}||| + \epsilon$. We claim that there exists a $t_1 \in (t_0, T]$ such that

$$v(t) \begin{cases} \leq & v(t_1), \qquad t < t_1, \\ \leq & |||v_{t_0}||| + \epsilon, \quad t = t_1 \end{cases} \tag{4.17}$$

and $v'(t_1) > 0$.

Note that

$$S = \{t \,|v(t) = |||v(t)||| \leq |||v_{t_0}||| + \epsilon, \;\; t \in [t_0, T]\} = \cup[\alpha_i, \beta_i], \alpha_{i+1} \geq \beta_i.$$

Then S is nonempty since $T \in S$. Thus by Lemma 4.1, $S = \cup[\alpha_i, \beta_i]$ and $\alpha_1 = infS$, we claim that

$$v(\alpha_1) = |||v_{\alpha_1}||| < |||v_{t_0}||| + \epsilon \tag{4.18}$$

In fact, in view of $|||v(t_0)||| < v(T) = |||v_{t_0}||| + \epsilon$, we choose T_1 such that

$$v(t) < v(T_1) = |||v_{t_0}||| + \epsilon/2, \quad t_0 \le t < T_1 < T.$$

Then $T_1 \in S$ and $T_1 \le \alpha_1$. This indicates that (4.18) is true.

Finally, we claim that there exists a $t_1 \in S$ such that $v'(t_1) > 0$. In fact, if it is not true, then $v'(t) \le 0, t \in S$, and so

$$\left\{ \begin{array}{lll} v(t) & = & v(\alpha_i), \quad t \in [\alpha_i, \beta_i]; \\ v(t) & \le & v(\beta_i) = v(\alpha_i) \quad t \in (\beta_i, \alpha_{i+1}). \end{array} \right.$$

Thus, $v(t) \le v(\alpha_1) < |||v_{t_0}||| + \epsilon, t \in S$ and hence $T \notin S$. This contradicts the hypothesis since it indicates the existence of t_1 such that $v'(t_1) > 0$.

On the other hand,

$$\begin{array}{lll} v'(t_1) & \le & -\alpha(t_1)v(t_1) + \beta(t_1)|||v_{t_1}||| \\ & \le & -\alpha(t_1)v(t_1) + \beta(t_1)v(t_1) \le 0. \end{array}$$

This contradicts with $v'(t_1) > 0$. Thus, (4.16) holds. Let $\epsilon \to 0$, we have

$$v(t) \le |||v_{t_0}|||.$$

The proof is completed.

Lemma 4.3 *Let $v(t)$ be a nonnegative, continuously differentiable function defined on $[t_0, b)$ such that*

$$v'(t) \le \alpha(t)v(t) + \beta(t) sup_{t-\tau \le s \le t} v(s),$$

for $t \in [t_0, b)$, where $\alpha(t) < 0$, $\beta(t) \ge 0$ are continuous functions and $\beta(t)$ is bounded. If $\alpha(t) + \beta(t) \le l < 0$, then there exist constants $\epsilon > 0$ such that

$$v(t) \le sup_{t_0-\tau \le s \le t_0} v(s) e^{-\epsilon t}$$

Proof: Since $\alpha(t) + \beta(t) \le l < 0$, there exist constant $\epsilon > 0$ such that

$$\alpha(t) + \beta(t)e^{\epsilon\tau} + \epsilon < 0.$$

Let $P(t) = v(t)e^{\epsilon t}$, we have

$$\begin{array}{lll} P'(t) & \le & \epsilon P(t) + [\alpha(t)v(t) + \beta(t) sup_{t-\tau \le s \le t} v(s)]e^{\epsilon t} \\ & \le & \epsilon P(t) + \alpha(t)P(t) + \beta(t)e^{\epsilon\tau} sup_{t-\tau \le s \le t} P(s) \end{array}$$

for $t \in [t_0, b)$.

Let $M = sup_{t_0-\tau\le s\le t_0} P(s)$, and for any $d > 1$, we shall prove that $P(t) < dM$ for $t \in [t_0, b)$. If it is not true, there exists a $t_1 \in (t_0, b)$ such that

$$P(t)\begin{cases} < & dM \quad t_0-\tau \le t < t_1 \\ = & dM \qquad t = t_1. \end{cases}$$

It is obviously that $P'(t_1) \ge 0$. On the other hand, it follows that

$$\begin{aligned} P'(t_1) &\le (\epsilon + \alpha(t_1))P(t_1) + e^{\epsilon\tau}\beta(t_1) sup_{t_1-\tau\le s\le t_1} P(s) \\ &\le (\epsilon + \alpha(t_1) + \beta(t_1)e^{\epsilon\tau})P(t_1) < 0. \end{aligned}$$

This contradiction shows that $P(t) < dM$ for all $t \in [t_0, b)$. Let $d \to 1$, it follows that $P(t) \le M$ for all $t \in [t_0, b)$ and hence

$$v(t) \le sup_{t_0-\tau\le s\le t_0} v(s)e^{-\epsilon t}, \quad t \in [t_0, b).$$

The proof is completed.

Theorem 4.1 *Assume that*

(i) there exist $n \times n$ matrices Q_i such that the solutions P_i of the systems

$$A^T(i)P_i + P_iA(i) + Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j = 0$$

are positive definite , $i = 1, 2, \cdots, N$;

(ii) $x^TP_ix \le y^TP_iy$ implies $||x|| \le ||y||$ for all $x, y \in R^n$ and P_i, $i = 1, 2, \cdots, N$;

(iii) there exist $l_1, l_2 > 0$, $m_1, m_2 \ge 0$ such that
$l_1x^Tx \le x^TP_ix \le l_2x^Tx$, and $||2x^TB(r(t))P_ix_t)|| \le m_1||x||^2 + m_2|||x_t|||^2$;

(iv) $-x^T(Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j - m_1I)x \le Kx^Tx$ and $l_2^{-1}K + l_1^{-1}m_2 \le 0$; where I is identity matrix and K is a constant.

The system (4.15) is then stable if $l_2^{-1}K + l_1^{-1}m_2 = 0$ and it is asymptotically stable if $l_2^{-1}K + l_1^{-1}m_2 < 0$.

Proof: Let

$$V_i(t, x, r(t)) = V_i = x_i^TP_ix_i, \quad for \;\; r(t) = i.$$

If

$$r(t) = \begin{cases} i, & t \in [t_{i_1}, t_{i_2}) \\ j, & t \in [t_{j_1}, t_{j_2}) = [t_{i_2}, t_{j_2}), \end{cases}$$

then $V_j(t,x,j)$ defined on interal $t \in [t_{j_1}, t_{j_2})$ can be extended naturally to the interval $[t_{j_1} - \tau, t_{j_2})$ as follows

$$V_j^*(t,x,j) = \begin{cases} V_j(t,x,j), & t \in [t_{j_1}, t_{j_2}); \\ V_j(t_{j_1},x,j), & t \in [t_{j_1} - \tau, t_{j_1}). \end{cases}$$

The derivative of V_i along with system (4.15) is

$$\begin{aligned} V_i' &= (x^T)'P_i x + x^T P_i x' \\ &= (x^T A^T(i) + x_t^T B^T(i))P_i x + x^T P_i(A(i)x + B(i)x_t) \\ &= x^T[A^T(i)P_i + P_i A(i)]x + 2x^T P_i B(i)x_t \\ &\leq -x^T[Q_i + \textstyle\sum_{j=1}^N \alpha_{ij}P_j]x + m_1||x||^2 + m_2|||x_t|||^2 \\ &\leq -x^T[Q_i + \textstyle\sum_{j=1}^N \alpha_{ij}P_j - m_1 I]x + m_2|||x_t|||^2 \\ &\leq Kx^T x + m_2|||x_t|||^2 \\ &\leq l_2^{-1}KV_i + l_1^{-1}m_2 sup_{t-\tau\leq s\leq t}V_i(s), \quad r(t) = i. \end{aligned}$$

If $l_2^{-1}K + l_1^{-1}m_2 = 0$, in view of Lemma 4.2, we have $x^T(t)P_i x(t) = V_i(t,x(t),i) \leq V_i(t_{i_1},x(t_{i_1}),i) = x^T(t_{i_1})P_i x(t_{i_1})$, $t \in [t_{i_1}, t_{i_2})$. According to the condition (ii), we have

$$||x(t)|| \leq ||x(t_{i_1})||, \quad t \in [t_{i_1}, t_{i_2}).$$

By induction, we can conclude

$$||x(t)|| \leq |||x_{t_0}|||, \quad t \in [t_0, \infty).$$

The stability of (4.15) is obtained. If $l_2^{-1}K + l_1^{-1}m_2 < 0$. Using Lemma 4.3, there exists a $\epsilon > 0$ such that

$$V_i(t,x(t),i) \leq sup_{t_{i_1}-\tau\leq s\leq t_{i_1}} V_i^*(t_{i_1},x(t_{i_1}),i)e^{-\epsilon(t-t_{i_1})},$$

Using the same approach for Theorem 3.1, we have

$$||x(t)|| \leq |||x_{t_0}|||e^{-\epsilon(t-t_0)} \quad t \in [t_0, \infty).$$

This implies system (4.15) is asymptotically stable. The proof is completed.

Now, let us consider the more general system

$$x(t)' = A(r(t))x(t) + g(t,x(t),x(t-\tau(t)),r(t)) \tag{4.19}$$

where $g(t,x,x_t,r(t))$ is continuous with respect to t, x, x_t and $g(t,0,0,r(t)) \equiv 0$.

Theorem 4.2 *Assume that*

(i) There exist $n \times n$ *matrices* Q_i *such that the solutions* P_i *of the systems*

$$A^T(i)P_i + P_iA(i) + Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j = 0$$

are positive definite $i = 1, 2, \cdots, N$*;*

(ii) $x^T P_i x \leq y^T P_i y$ *implies* $||x|| \leq ||y||$ *for all* $x, y \in R^n$ *and* P_i, $i = 1, 2, \cdots, N$*;*

(iii) there exist $l_1, l_2 > 0$, $m_1(t), m_2(t) \in C(R, R_+)$ *such that* $l_1 x^T x \leq x^T P_i x \leq l_2 x^T x$ *and* $||2g^T(t, x, x(t-\tau(t)), r(t))P_i x|| \leq m_1(t)||x||^2 + m_2(t)|||x_t|||^2$*;*

(iv) $-x^T(Q_i + \sum_{j=1}^{N} \alpha_{ij}P_j - m_1(t)I)x \leq K(t)x^T x$ *and* $l_2^{-1}K(t) + l_1^{-1}m_2(t) \leq 0$*; where* I *is identity matrix and* $-K \in C([t_0, \infty), R^+)$.

Then system (4.19) *is stable if* $\int_{t_0}^{\infty} K(s)ds \geq -\infty$ *and it is asymptotically stable if* $\int_{t_0}^{\infty} K(s)ds = -\infty$.

Proof: Let

$$V_i(t, x, r(t)) = V_i = x_i^T P_i x_i, \quad for \;\; r(t) = i.$$

then, the derivative of V_i along with system (4.19) is

$$\begin{aligned}
V_i' &= (x^T)'P_i x + x^T P_i x' \\
&= (x^T A^T(i) + g^T(t, x, x(t-\tau(t)), r(t)))P_i x \\
&+ \; x^T P_i(A(i)x + g(t, x, x(t-\tau(t)), r(t))) \\
&= x^T[A(i)^T P_i + P_i A(i)]x + 2x^T P_i g(t, x, x(t-\tau(t)), r(t)) \\
&= -x^T[Q_i + \textstyle\sum_{j=1}^{N} \alpha_{ij}P_j]x + m_1(t)||x||^2 + m_2(t)|||x_t|||^2 \\
&\leq -x^T[Q_i + \textstyle\sum_{j=1}^{N} \alpha_{ij}P_j - m_1(t)I]x + m_2(t)|||x_t|||^2 \\
&\leq K(t)x^T x + m_2(t)|||x_t|||^2 \\
&\leq l_2^{-1}K(t)V_i + l_1^{-1}m_2(t) sup_{t-\tau \leq s \leq t} V_i(s), \quad r(t) = i.
\end{aligned}$$

If $l_2^{-1}K(t) + l_1^{-1}m_2(t) \leq 0$, similar to Theorem 4.2, we can get the conclusion

$$||x(t)|| \leq |||x_{t_0}|||, \quad t \in [t_0, \infty).$$

The stability of (4.19) is obtained. If $l_2^{-1}K(t) + l_1^{-1}m_2(t) < l < 0$. Using Lemma 4.3 and the proof of Theorem 4.2, we can obtain

$$||x(t)|| \le |||x_{t_0}|||e^{-\epsilon(t-t_0)} \quad t \in [t_0, \infty).$$

Thus system (4.19) is asymptotically stable. The proof is therefore complete.

Example 4.1 *Consider the jump system*

$$x(t)' = A(r(t))x(t) + g(t, x(t), x(t-\tau), r(t)) \tag{4.20}$$

where $x(t) \in R^2$,

$$A(r(t))x(t) + g(t, x(t), x(t-\tau), r(t)) =$$

$$\begin{pmatrix} -6 & 1 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} sinx_1(t-\tau) \\ cosx_2(t-\tau) \end{pmatrix},$$

$$for \;\; r(t) = 1;$$

$$A(r(t))x(t) + g(t, x(t), x(t-\tau), r(t)) =$$

$$\begin{pmatrix} -5 & 0 \\ 1 & -7 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} ln(1 + |x_2(t-\tau)|) \\ 0 \end{pmatrix},$$

$$for \;\; r(t) = 2.$$

Without loss of generality, let $\alpha_{ij} = 1/2, 1 \le i, j \le 2$. *We have*

$$A(1) = \begin{pmatrix} -6 & 1 \\ 2 & -4 \end{pmatrix}; \; A(2) = \begin{pmatrix} -5 & 0 \\ 1 & -7 \end{pmatrix};$$

$$g(t, x(t), x(t-\tau), 1) = \begin{pmatrix} sinx_1(t-\tau) \\ cosx_2(t-\tau) \end{pmatrix};$$

$$g(t, x(t), x(t-\tau), 2) = \begin{pmatrix} ln(1 + |x_2(t-\tau)|) \\ 0 \end{pmatrix}.$$

Take

$$Q_1 = \begin{pmatrix} 11 & -3 \\ -3 & 7 \end{pmatrix}; \; Q_2 = \begin{pmatrix} 9 & -1 \\ -1 & 13 \end{pmatrix}.$$

Then $P_1 = P_2 = I$ *and*

$$||2g_1^T P_1 x|| \le ||x||^2 + |||x_t|||^2;$$
$$||2g_2^T P_2 x|| \le ||x||^2 + |||x_t|||^2,$$

for $l_1 = l_2 = 1$, $m_1(t) = m_2(t) = 1$ *and* $K(t) = -5$. *It is easy to see that all the conditions of Theorem 4.2 are satisfied and hence, system* (4.20) *is asymptotically stable.*

5 Conclusions

In this paper, we have studied the stability issues of a class of hybrid dynamical systems, namely, dynamic systems with jump parameters. Our approach has utilized the Lyapunov functions and delay differential inequalities for the stability criteria establishment. The stability results may be generalized to other hybrid dynamic systems.

Acknowledgment

This research was supported by the Natural Sciences and Engineering Research Council of Canada under grants No. OGPIN108310 and No. RGPIN203560.

References

[1] B.D.O. Anderson and J.B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, (1971).

[2] M. Mariton, *Jump Linear System in Automatic Control*, Marcel Dekker, New York and Basel, (1990).

[3] D.D. Sworder, *Feedback Control of a Class of Linear Systems with Jump Parameters*, IEEE Trans. Automatic Control, Vol.AC-14, pp.9-14, (1969).

[4] Y. Ji and J.C. Howard, *Controllability, Stabilizability, and Continuous-Time Markovizn Jump Linear Quadratic Control*, IEEE Trans. Auto. Control, Vol.35, pp.777-788 (1990).

[5] Z. Gajic, D. Petkovski, and N. Harkara, *Lyapunov Iterations for Optimal Control of Jump Linear Systems at Steady State*, IEEE Trans. Auto. Control, Vol.40, No.11, pp.1971-1975 (1995).

[6] X. Shen, Q. Xia, M. Rao and Y. Ying, *Near-optimum Regulators for Singularly Perturbed Jump Systems*, Control Theo. and Advan. Tech. Vol.9, No.3, pp.759-773, (1993).

[7] X. Shen and V.G. Gourishankar, Q. Xia and M. Rao, *Composite Controller Design for Discrete Singularly Perturbed Systems with Random Parameters*, J. Franklin Institute, Vol.331B, No.2, pp.217-227, (1994).

[8] W.P. Blair, Jr and D.D. Sworder, *Feedback Control of a Class of a Linear Discrete Systems with Jump Parameters and Quadratic Cost Criteria*, Int. J. Control, Vol.21, pp.833-841, (1975).

[9] M. Mariton, *On Controllability of Linear Systems with Stochastic Jump Parameters*, IEEE Trans. Auto. Control, Vol.AC-31, pp.680-683,(1986).

[10] H.J. Chizeck, A.S. Willsky and D. Castanon, *Discrete-time Markovian-jump Linear Quadratic Optimal Control*, Int. J. Control, Vol.43, pp.213-231, (1986).

[11] P. Caines and J. F. Zhang, *On the Adaptive Control of Jump Parameter Systems via Non-linear Filtering*, SIAM. J. Control Optim., Vol.33, No.6, pp.1758-1777 (1995).

Received August 2000; revised January 2001.
http://monotone.uwaterloo.ca/~journal