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CHAOS SYNCHRONIZATION USING IMPULSIVE DRIVING AND APPLICATIONS TO SECURE COMMUNICATIONS

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Abstract. This paper investigates chaos synchronization using impulsive driving. A stability result is established first for a class of impulsive differential equations. This result is then used to obtain synchronization criterion for chaotic systems. As an application, a chaotic communication scheme is developed. Simulation results show that our scheme has advantages over some other schemes found in the literature.

Keywords. Chaos, synchronization, impulsive differential equations, impulsive driving, secure communication.

1 Introduction

Since Pecora and Carrol [8] proposed their well-known self synchronization method to synchronize two chaotic systems, several variations of the method have been successfully established such as synchronization by cascaded drive-response [1], synchronization by partial replacement [5], synchronzation by unidirectional coupling [8], and synchronization by sporadic driving [9]. Each method has its own constraint in applications. For example, self-synchronization requires that the driven subreponse system is asymptotically stable and synchronization by sporadic driving requires that the undriven subsystem is asymptotically stable. To decompose a practical system into subsystems which satisfy such requirement is not an easy problem in some situations. In addition, it is impossible to recover the original information signal exactly from the chaotic masked signal if self-synchronization method is used in a chaotic masking communication scheme. This is because the driving signal is acting as perturbation to the driven subreponse system, and chaotic systems are inherent sensitive to perturbations. Synchronization by impulsive driving has some advantages over the method of Pecora and Carrol and its variations in the sense that it is not necessary to decompose a chaotic system into subsystems and only a small portion of channel resource is needed to transmit the driving impulses. Impulsive synchronization of Lorenz systems was reported in [10].

In this paper, we investigate the synchronization of chaotic systems by impulsive driving in a more general way. The stability criterion for the error system of two very general uni-directionally coupled identical nonlinear systems, which can be described by impulsive differential equations, is established. Then the theoretical result is used to study the synchronization of two chaotic systems such as Chen's systems [2], chemical reaction systems [4], and hysteresis systems [8]. The simulation result of synchronizing two Chen's systems by using partial replacement is also provided. For the general theory of impulsive differential equations, see [6].

The organization of this paper is as follows. In section 2, some definitions and impulsive differential equations are introduced. In section3, a stability criterion for a general impulsive differential equation is estiblished. In section 4, the problem formulation is given. In section 5, simulation results for the chosen systems are presented. In section 6, the concluding remarks are provided.

2 Preliminaries

Consider the impulsive differential system

$$\begin{cases} x' = f(t,x), & t \neq t_k \\ \Delta x = I_k(x), & t = t_k, \quad k \in N \\ x(t_0^+) = x_0 \end{cases} \tag{2.1}$$

where $\Delta x(t_k) = x(t_k^+) - x(t_k)$, $x(t_k^+) = \lim_{t\to t_k^+} x(t)$, $t_1 < t_2 < ...$, and $lim_{t\to\infty} t_k = \infty$.

We assume that the functions $f(t,x)$ and $I_k(x)$ in (2.1) satisfy all the required conditions [6] so that all solutions of (2.1) starting at (t_0, x_0), denoted by $x(t) = x(t,t_0,x_0)$, exist for all $t > t_0$. We also assume that $f(t,0) = 0$ and $I_k(0) = 0$ for all k so that we have the trivial solution of (2.1).

To facilitate the discussion, it is convenient to introduce the following notations and concepts.

$PC = \{\sigma : R_+ \to R$: continuous on $(t_{k-1}, t_k]$ and $\lim_{t\to t_k^+} \sigma(t) = \sigma(t_k^+)$ exists for all $k \in N\}$.

$K = \{\phi : R_+ \to R_+$: continuous on R_+, strictly increasing and $\phi(0) = 0\}$.

$K_0 = \{\psi : R_+ \to R_+$: continuous on R_+, $\psi(s) > 0$ if $s > 0$ and $\psi(0) = 0\}$.

$v_0 = \{V : R_+ \times R^n \to R_+$: continuous on $(t_{k-1}, t_k] \times R^n$, locally Lipschitz in x and $V(t_k^+, x)$ exists for each $k \in N\}$.

$S(\rho) = \{x \in R^n : ||x|| < \rho\}$.

Definition 2.1 *For $(t,x) \in (t_{k-1}, t_k] \times R^n$, we define*

$$D^+V(t,x) = \lim_{h\to 0} sup \frac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)]. \tag{2.2}$$

Definition 2.2 *System* (2.1) *is called stable if for any given* $\epsilon > 0$ *and* $t_0 \in R_+$*, there exists a* $\delta = \delta(\epsilon, t_0) > 0$ *such that* $||x_0|| < \delta$ *implies that* $||x(t, t_0, x_0)|| < \epsilon$ *for all* $t \geq t_0$*.*

Definition 2.3 *System* (2.1) *is called asymptotically stable if it is stable and for any solution* $x(t) = x(t, t_0, x_0)$ *of (2.1), there exists a* $\sigma > 0$ *such that* $||x_0|| < \sigma$ *implies that* $lim_{t\to\infty}||x(t)|| = 0$*.*

We shall need the following lemma [7].

Lemma 2.1 *Assume that*

(i) there exist constants ρ, ρ_0 *with* $0 < \rho_0 < \rho$ *such that* $x \in S(\rho_0)$ *implies that* $x + I_k(x) \in S(\rho)$*, for all* $k = 1, 2, ...,$

(ii) there exist $a, b \in \kappa$ *such that for* $V(t, x) \in v_0$*,* $b(||x||) \leq V(t, x) \leq a(||x||)$ *on* $\mathrm{R}_+ \times S(\rho)$*, and there exists* $\Psi_k \in \kappa_0$ *such that*
$V(t_k^+, x + I_k(x)) \leq \Psi_k(V(t_k, x))$*, for all* $k = 1, 2, ...,$

(iii) there exist $c \in \kappa$ *and* $p \in PC$ *such that*
$D^+V(t, x) \leq p(t)c(V(t, x)), x \in S(\rho), t \neq t_k,$

(iv) there exists constant $\sigma > 0$ *such that* $\forall z \in (0, \sigma)$
$\int_{t_k}^{t_{k+1}} p(s)ds + \int_z^{\Psi_k(z)} \frac{ds}{c(s)} \leq -\gamma_k$*, for some constant* γ_k *and* $k = 1, 2, ...$

Then the trivial solution of system (2.1) *is stable if* $\gamma_k \geq 0$ *for all* $k = 1, 2, ...,$ *and asymptotically stable if, in addition,* $\sum_{k=1}^{\infty} \gamma_k = \infty$*.*

3 Stability Criterion

Assume that the vector field $f(t, x)$ in system (2.1) can be decomposed into the sum of a linear part Ax and a nonlinear part $R(t, x)$, then we have the following system

$$\begin{cases} x' = Ax + R(t, x), & t \neq t_k \\ \Delta x = Bx, & t = t_k \\ x(t_0^+) = x_0, & k = 1, 2, ... \end{cases} \tag{3.3}$$

where A and B are $n \times n$ matrices and $R \in C[R_+ \times S(\rho), R^n]$.

We shall establish, in this section, sufficient conditions to guarantee the asymptotic stability of system (3.3).

Let G be a positive definite $n \times n$ matrix. Define Q and P by

$$Q = GA + A^T G \qquad \textit{and} \qquad P = (I + B^T)G(I + B), \tag{3.4}$$

where I is the identity matrix.

Let β and η be the largest eigenvalues of Q and P, respectively; λ_1 and λ_2 be the smallest and largest eigenvalues of G, respectively. Then we have the following result.

Theorem 3.1 *Assume that there exists a positive definite matrix G such that*

(i) for some $\theta \in C[R, R]$,

$$x^T GR(t,x) \leq \theta(t) x^T Gx, (t,x) \in R \times R^n; \tag{3.5}$$

(ii) for some constants γ_k, $k = 1, 2, ...$,

$$\frac{\beta}{\lambda_1}(t_{k+1} - t_k) + \int_{t_k}^{t_{k+1}} \theta(s) ds + \ln \frac{\eta}{\lambda_1} \leq -\gamma_k; \tag{3.6}$$

(iii) the spectral radius of $I + B$ satisfies $\rho(I + B) \leq 1$.

Then the trivial solution of system (3.3) *is stable if $\gamma_k \geq 0$ for all $k = 1, 2, ...$, and asymptotically stable if, in addition, $\sum_{k=1}^{\infty} \gamma_k = \infty$.*

Proof: Choose the Lyapunov function $V(t,x) =< Gx, x >$. Then if $t \neq t_k$, we have

$$\begin{aligned} D^+V(t,x) &= (x^T Gx)' \\ &= (x^T)' Gx + x^T Gx' \\ &= [Ax + R(t,x)]^T Gx + x^T G' x + x^T G[Ax + R(t,x)] \\ &= x^T [GA + A^T G] x + R^T(t,x) Gx + x^T GR(t,x) \\ &= x^T Qx + 2x^T GR(t,x) \\ &\leq [\beta/\lambda_1 + \theta(t)] V(t,x) \\ &= p(t) V(t,x) \end{aligned}$$

where $p(t) = \beta/\lambda_1 + \theta(t)$.

When $t = t_k$, we have

$$\begin{aligned} V(t^+, x + Bx) &= (x + Bx)^T G(x + Bx) \\ &= x^T (I + B^T) G(I + B) x \\ &\leq \eta/\lambda_1 x^T Gx \\ &= \mu V(t,x) \end{aligned}$$

where $\mu = \eta/\lambda_1$.

Also, we have

$$\lambda_1 |x|^2 \leq V(t,x) \leq \lambda_2 |x|^2 \tag{3.7}$$

and

$$||x + Bx|| \leq ||I + B|| ||x|| \leq ||x|| \tag{3.8}$$

which implies that $x + Bx \in S(\rho)$.

The inequality (3.6) is equivalent to $\int_{t_k}^{t_{k+1}} p(s) ds + \int_z^{\Psi_k(z)} \frac{ds}{c(s)} \leq -\gamma_k$ if we take $\Psi_k(z) = \mu z$ and $c(s) = s$. Obviously, all the conditions in Lemma 2.1 are satisfied. Thus, the proof of Theorem 3.1 is complete.

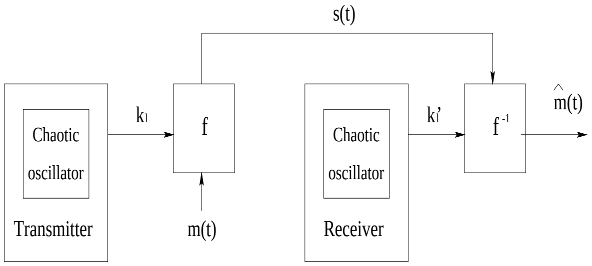


Figure 1: A general chaotic masking communication scheme

If $p(t)$ is independent of t, say $p(t) = \beta/\lambda_1 + \theta = \xi$, then from the inequality (3.6) we can get the maximum time interval between two consecutive time instants t_k and t_{k+1} as

$$\Delta_{max} = max\{\Delta t_k\} = \lim_{\alpha \to 1^+} \left| \frac{\ln \alpha\mu}{\xi} \right| \tag{3.9}$$

where $\alpha = e^{\gamma_k} > 1$.

It is important to get the estimation of the maximum time interval between two time instants to ensure that the chaos synchronization can be achieved by impulsive driving. It is also important to know how to construct the matrix B in system (3.3). The condition $\rho(I + B) \leq 1$ in Theorem 3.1 and the expression (3.9) provided us a unified way to choose the matrix B and the time separation of impulsive time instants.

4 Synchronizing chaos for communication

The general idea of chaotic communication scheme is to transmit an information signal with a broadband chaotic carrier signal and to use synchronization to recover the information signal at the receiver. Several implementations of this general concept have been proposed such as chaotic masking, chaos modulation, chaos shift keying, parameter modulation, and encoding using generalized synchronization, to name a few.In this paper, we consider a chaotic masking communication scheme by impulsive driving. For chaotic masking communication scheme by self synchronization, see [3].

We introduce a general chaotic masking communication scheme as shown in Figure 1. The masked signal $s(t)$ is obtained by using an encrypting mapping f to hide the information signal $m(t)$ at the transmitter end, i.e., $s(t) = f_{k_l}(m(t))$, where the key $k_l = (s_1, ..., s_l)$ is a vector of the state varialbes of the chaotic system at the transmitter end and $1 \leq l \leq n$. This masked signal is then transmitted through the channel. At the receiver end, the information signal is recovered by applying the decrypting mapping f^{-1}, the reverse of f, to $\hat{s}(t)$ (the estimation of $s(t)$), i.e., $\hat{m}(t) = f_{k'_l}^{-1}(\hat{s}(t))$, where the key $k'_l = (s'_1, ..., s'_l)$ is the corresponding vector of the state varialbes of the chaotic system at the receiver end and $1 \leq l \leq n$. If the two chaotic systems at the transmitter end and receiver end are synchronized, i.e., $k'_l = k_l$, and $s(t)$ can be estimated exactly, i.e., $\hat{s}(t) = s(t)$, then the information signal $m(t)$ can be recovered definitely. To simplify the illustrations, we use a very simple encrypting mapping and the corresponding decrypting mapping in the sequel.

We first recall the method of self synchronization. An autonomous n dimensional dynamical system

$$u' = f(u) \tag{4.10}$$

can be decomposed arbitrarily into two subsystems as

$$v' = g(v, w) \tag{4.11}$$

and

$$w' = h(v, w) \tag{4.12}$$

where $u = (v, w)^T$.

We create a response subsystem, a copy of subsystem (4.12), with the set of variables v are substituted for the corresponding $\tilde{v}$ in h as

$$\tilde{w}' = h(v, \tilde{w}). \tag{4.13}$$

We say that the system (4.10) has the property of self synchronization if $\lim_{t\to\infty} e = \lim_{t\to\infty}(w - \tilde{w}) = 0$ holds.

In the drive-reponse scenario, one of the dynamical variables in the reponse is replaced completely with its counterpart from the drive. We can also replace a response variable with the drive counterpart only in certain locations. A chaotic masking communication scheme by using partial replacement, which the variable y' in the chaotic receiver is replaced by its counterpart y in the chaotic tranmitter plus the information signal $m(t)$, is shown in Figure 2.

The masked signal $s(t)$ is obtained by adding the information signal $m(t)$ and the state variable y of the chaotic system at the transmitter end, i.e., $s(t) = f_{(y)}(m(t)) = m(t) + y$. If the two chaotic systems at the transimitter and receiver ends can are synchronized and the received signal has no distortion, then the information signal is recovered by subtracting the variable y' from $s(t)$, i.e., $\hat{m}(t) = f_{(y')}(s(t)) = s(t) - y' = m(t) + y - y'$. However, because the coupling signal acts as perturbation to the response subsystem, it may destroy the synchronization between two coupled chaotic systems.

To overcome this problem, We introduce the chaotic synchronization by impulsive driving. Let us consider the following two chaotic systems

$$x' = Ax + r(x) \tag{4.14}$$

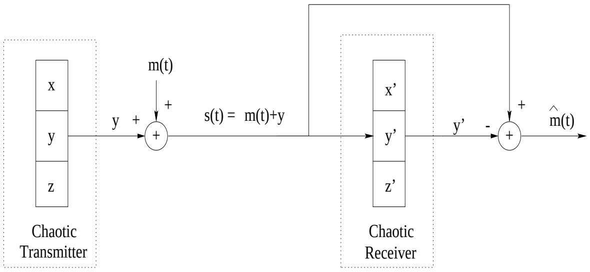


Figure 2: Chaotic masking communication scheme using partial replacement synchronization

and

$$\begin{cases} y' = Ay + r(y), & t \neq t_k \\ \Delta y = -B(x - y), & t = t_k, \quad k = 1, 2, ... \end{cases} \tag{4.15}$$

where x and y are n dimensional variables, A and B are $n \times n$ matrices, $r(x)$ and $r(y)$ are nonlinear functions, and the time instants sequence $\{t_k\}_{k=1}^{\infty}$ satisfies that $t_1 < t_2 < ... < t_k < ...$, and $\lim_{t\to\infty} t_k = \infty$.

We say that the system (4.14) and the system (4.15) can be synchronized by impulsive driving if the error system

$$\begin{cases} e' = Ae + \phi(x, y), & t \neq t_k \\ \Delta e = Be, & t = t_k, \quad k = 1, 2, ... \end{cases} \tag{4.16}$$

where $e^T = (x_1 - y_1, x_2 - y_2, x_3 - y_3)$, is asymptotically stable.

From the stability criterion in Section 3, we get the following criterion on chaos synchronization by impulsive driving.

Corollary 4.1 *System* (4.14) *and system* (4.15) *are synchronized if* $\rho(I + B) \leq 1$ *and* $0 \leq \Delta t_k \leq \Delta_{max} - \gamma_k$*, where* Δ_{max} *is defined in* (3.9) *and* $\sum_{k=1}^{\infty} \gamma_k = \infty$*.*

A chaotic masking communication scheme using synchronization by impulsive driving can be illustrated by Figure 3.

In this scenario, the switch S_1 at the transimitter end is used to control the time instants $\{\tilde{t}_k\}_{k=1}^{\infty}$ at which a sequence of discrete state vectors is transimitted to the receiver, the switch S_2 at the receiver end is used to control the time instants $\{t_k\}_{k=1}^{\infty}$ at which a sequence of discrete state vectors is obtained, and the switch S_3 acting the same as S_2 is used to get the difference e of the two discrete state vectors. It is not necessary that the switches S_1 and S_2 act simultaneously because only the time interval between two consecutive impulses is the key element to synchronize the two chaotic systems, but not the absolute time instants. Thus, if τ is denoted the impulse transmission delay plus the processing time at the receiver end, then it is

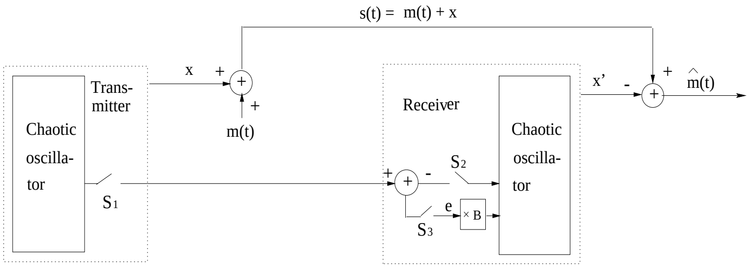


Figure 3: Chaotic masking communication scheme using synchronization by impulsive driving

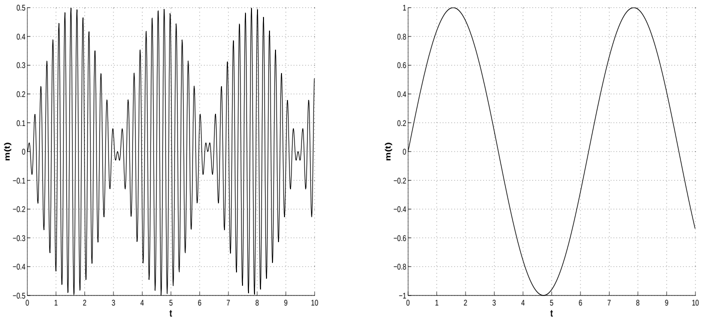
sufficient to guarantee the synchronization of the two chaotic systems when $\{\tilde{t}_k\}_{k=1}^{\infty}$ and $\{t_k\}_{k=1}^{\infty}$ satisfy the relation $\tilde{t}_k = t_k - \tau$. The masked signal is obtained by the same method as in the self synchronization communication scheme. However, in this communication scheme the driving signal is not the masked signal $m(t)+x$, but the impulses obtained by the weighted differences of the corresponding variables at the transmitter end and the receiver end. The information signal can be recovered exactly if the two chaotic systems are synchronized. From Section 3, we know that the chaos synchronization can be guaranteed by impulsive driving. In addition, the masked signal $m(t) + x$ would not destroy the synchronization because it does not act as perturbation in this case.

5 Simulation results

In this section, we use the stability criterion given in section 3 to synchronize two chaotic systems in a chaotic masking communication scheme. We give examples by using Chen's system, chemical reaction systems, and hysteresis systems as chaotic oscillators. To illustrate the ability of synchronization by using impulsive driving, the simulation result of using Chen's system by partial replacement is provided first, and then the simulation results of using Chen's system, a chemical reaction system, and a hysteresis system by impulsive driving are provided.

In the following examples, the information signal $m(t) = 0.5sin(30t)sin(t)$, shown in Figure 4(a), is used in Examples 5.1 and 5.2, and the information signal $m(t) = sin(t)$, shown in Figure 4(b), is used in Examples 5.3 and 5.4.

Example 5.1 *Chaotic masking communication by synchronizing two Chen's*



(a) (b)

Figure 4: Information signals: (a) $m(t) = 0.5sin(30t)sin(t)$, (b) $m(t) = sin(t)$

systems with partial replacement. Chen's system can be described by

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(R - \sigma - z) + Ry \\ \dot{z} = xy - Bz, \end{cases} \tag{5.17}$$

where σ, R, B *are constants. In the simulation, the chaotic system at the transmitter end is given by*

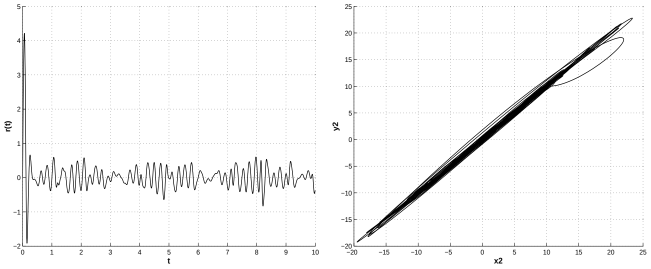
$$\begin{cases} \dot{x}_1 = \sigma(x_2 - x_1) \\ \dot{x}_2 = x_1(R - \sigma - x_3) + Rx_2 \\ \dot{x}_3 = x_1 x_2 - Bx_3, \end{cases} \tag{5.18}$$

and the chaotic system at the receiver end is given by

$$\begin{cases} \dot{y}_1 = \sigma(y_2 - y_1) \\ \dot{y}_2 = y_1(R - \sigma - y_3) + Rs \\ \dot{y}_3 = y_1 y_2 - By_3, \end{cases} \tag{5.19}$$

where s *is the chaotic masked signal. By taking* $(\sigma, R, B) = (35, 28, 3)$*, the initial points* $x_0^T = (x_{10}, x_{20}, x_{30}) = (10, 10, 10)$ *at the transmitter end,* $y_0^T = (y_{10}, y_{20}, y_{30}) = (10.5, 10, 20)$ *at the receiver end, a fourth order Runge-Kutta method with step size* $dt = 0.002$ *and evolution time* $T = 10$ *is used in the simulation, and the results are shown in Figure 5.*

It can be seen that the information signal can be recovered, but is not the same as the original signal. This is because that the transmitted signal s is acting as perturbation to the chaotic oscillator at the receiver end so that the synchronization can not be achieved exactly.



(a) (b)

Figure 5: Simulation result by using Chen's systems with partial replacement synchronization: (a) the recovered signal, (b) synchronization between x_2 and y_2

Example 5.2 *Chaotic masking communication by synchronizing two Chen's systems with impulsive driving. We rewrite Chen's system at the transmitter end as*

$$x' = Ax + r(x), \tag{5.20}$$

where $x^T = (x_1, x_2, x_3)$,

$$A = \begin{pmatrix} -\sigma & \sigma & 0 \\ R-\sigma & R & 0 \\ 0 & 0 & -B \end{pmatrix},$$

and $r^T(x) = (0, -x_1x_3, x_1x_2)$. *The impulsive form at the receiver end corresponds to*

$$\begin{cases} y' = Ay + r(y) & t \neq t_k \\ \Delta y = -B(x-y), & t = t_k, \quad k = 1, 2, ... \end{cases} \tag{5.21}$$

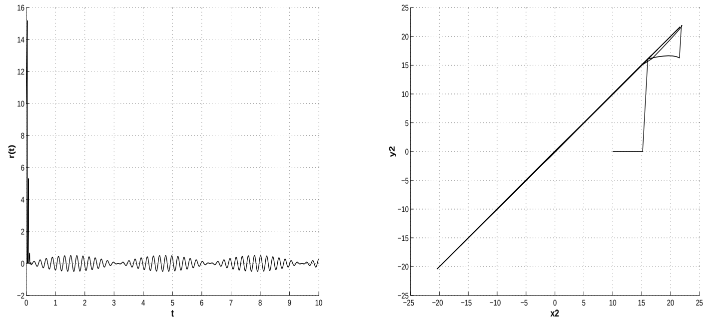
where $y^T = (y_1, y_2, y_3)$ *and* $r^T(y) = (0, -y_1y_3, y_1y_2)$. *To guarantee these two systems to be synchronized, the following error system should be asymptotically stable.*

$$\begin{cases} e' = Ae + \phi(x, y) & t \neq t_k \\ \Delta e = Be, & t = t_k, \quad k = 1, 2, ... \end{cases} \tag{5.22}$$

where $e^T = (e_1, e_2, e_3) = (x_1 - y_1, x_2 - y_2, x_3 - y_3)$, *and* $\phi^T(x, y) = (0, -(x_1x_3 - y_1y_3), (x_1x_2 - y_1y_2))$.

From $\rho(I + B) \leq 1$, *we can choose* B *as*

$$B = \begin{pmatrix} -0.95 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$



(a) (b)

Figure 6: Simulation result by using Chen's systems with impulsive driving synchronization: (a) the recovered signal, (b) synchronization between x_2 and y_2

Let $V(e) = < Ge, e >$ *with* $G = I$ *be a Lyapunov function. Obviously, 1 is the maximum and also the minimum eigenvalue of* G*, i.e.,* $\lambda_1 = \lambda_2 = 1$*. With* $(\sigma, R, B) = (35, 28, 3)$*, the largest eigenvalue of* $Q = GA + A^T G = A + A^T$ *is* $\beta \doteq 61.9420$*. The largest eigenvalue of* $(I + B^T)G(I + B) = (I + B^T)(I + B)$ *is* $\eta = 0.0025$*.*

Since the amplitudes of chaotic signals are bounded above we can assume the maximum value is M*. In our example, the amplitudes of state variables do not exceed* 40*, so we can choose* $M = 40$*.*

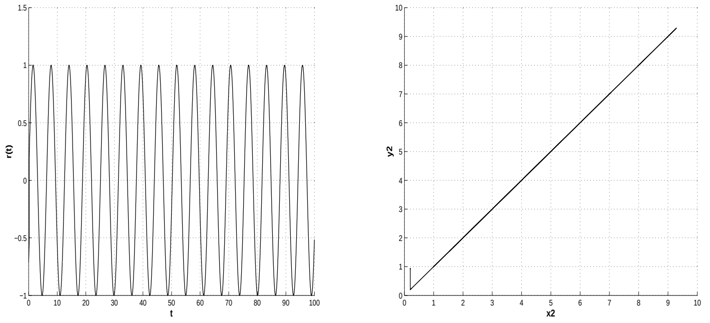
On the other hand,

$$\begin{aligned}
e^T\phi(x,y) &= -x_1x_3e_2 + y_1y_3e_2 + x_1x_2e_3 - y_1y_2e_3 \\
&= (-x_1x_3e_2 + x_1y_3e_2) + (-x_1y_3e_2 + y_1y_3e_2) \\
&\quad +(x_1x_2e_3 - x_1y_2e_3) + (x_1y_2e_3 - y_1y_2e_3) \\
&= -y_3e_1e_2 + y_2e_1e_3 \\
&\leq |y_3|(e_1^2 + e_2^2)/2 + |y_2|(e_1^2 + e_3^2)/2 \\
&= [|y_3| + |y_2|)e_1^2 + |y_1|e_2^2 + |y_2|e_3^2]/2 \\
&\leq Me^Te \\
&\leq Me^TGe \\
&= MV(t,x)
\end{aligned}$$

Thus, from (3.9)*, we get the maximum time interval of time instants as follows.*

$$\begin{aligned}
\Delta_{max} &= \lim_{\alpha\to 1^+} |ln(\alpha\mu)/\xi| \\
&= \lim_{\alpha\to 1^+} |ln(\alpha\eta/\lambda_1)/(\beta/\lambda_1 + 2M)| \\
&\doteq 0.06.
\end{aligned}$$

With $\Delta = 0.04$*,* $x_0^T = (x_{10}, x_{20}, x_{30}) = (10, 10, 10)$ *and* $y_0^T = (y_{10}, y_{20}, y_{30}) =$



(a) (b)

Figure 7: Simulation result by using the chemical reaction systems with impulsive driving synchronization: (a) the recovered signal, (b) synchronization between x_2 and y_2

$(0,0,0)$*, a fourth order Runge-Kutta method with step size* $dt = 0.005$ *and evolution time* $T = 10$*, the simulation results are shown in Figure 6. It can be seen that the original signal is recovered exactly.*

In general, self synchronization is very sensitive to the frequencies of information signal, especially when the information signal has lower frequencies. To make the recovered signal recognizable, an information signal with much higher frequency component is used in the above two examples. In the following examples, we use a much lower frequency signal as the information signal to demonstrate further that there is no such a problem if we use chaos synchronization by impulsive driving.

Example 5.3 *Chaotic masking communication by synchronizing two chemical reaction systems with impulsive driving. The chemical reaction system can be described as*

$$\begin{cases} \dot{x} = pw/(1+w^{10}) - 0.1x \\ \dot{y} = 0.1x - 0.2yz \\ \dot{z} = 0.2z(y-w) \\ \dot{w} = 0.2zw - 0.1w \end{cases} \tag{5.23}$$

where p *is a constant.*

Using the same method as we did in Example 5.2, we can get the system at the transmitter end with respect to x *and the corresponding impulsive system at the receiver end with respect to* y*. Furthermore, we can get the error system as*

$$\begin{cases} e' = Ae + \phi(x,y), & t \neq t_k \\ \Delta e = Be, & t = t_k, \quad k = 1,2,... \end{cases} \tag{5.24}$$

where

$$\mathbf{A} = \begin{pmatrix} -0.1 & 0 & 0 & p \\ 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \end{pmatrix},$$

$e^T = (e_1, e_2, e_3, e_4) = (x_1 - y_1, x_2 - y_2, x_3 - y_3, x_4 - y_4)$, *and* $\phi^T(x, y) = (p[x_4/(1 + x_4^{10}) - [y_4/(1 + y_4^{10})], -0.2(x_2x_3 - y_2y_3), 0.2(x_2x_3 - x_3x_4 - y_2y_3 + y_3y_4), 0.2(x_3x_4 - y_3y_4))$.

Again, we can take

$$\mathbf{B} = \begin{pmatrix} -0.95 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We choose $V(e) =< Ge, e >$ *with* $G = I$ *as a Lyapunov function. If we take* $p = 2.5$, *then the largest eigenvalue of* $Q = GA + A^TG = A + A^T$ *is* $\beta \doteq 2.3022$. *The largest eigenvalue of* $(I + B^T)G(I + B) = (I + B^T)(I + B)$ *is* $\eta = 0.0025$. *Let* M *be the maximum amplitude of the state variable amplitudes. Let* $K = max(0.8M, p/2 + 0.5M)$. *Then we have*

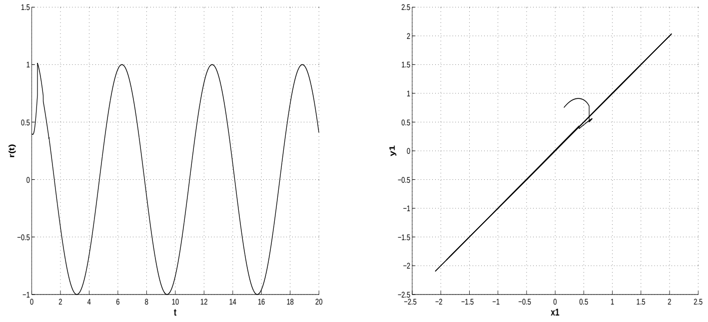
$$\begin{aligned} e^T\phi(x, y) &= pe_1[x_4/(1 + x_4^{10}) - y_4/(1 + y_4^{10})] - 0.2e_2(x_2x_3 - y_2y_3) \\ &\quad +0.2e_3(x_2x_3 - x_3x_4 - y_2y_3 + y_3y_4) + 0.2e_4(x_3x_4 - y_3y_4) \\ &= p\tfrac{1-\sum_{i=0}^{8} x_4^{i+1}y_4^{9-i}}{(1+x_4^{10})(1+y_4^{10})}e_1e_4 - 0.2x_3e_2^2 + 0.2(x_3 - y_2)e_2e_3 + 0.2(y_2 - x_4)e_3^2 + \\ &\quad +0.2(y_4 - y_3)e_3e_4 + 0.2x_3e_4^2 \\ &\leq p/2e_1^2 + (0.3|x_3| + 0.1|y_2|)e_2^2 + (0.1|x_3| + 0.3|y_2| + 0.1|y_3| + 0.3|y_4|)e_3^2 \\ &\quad +(p/2 + 0.3|x_4| + 0.1|y_3| + 0.1|y_4|)e_4^2 \\ &\leq Ke^Te \\ &\leq Ke^TGe \\ &= KV(t, x) \end{aligned}$$

In this example, the amplitudes of state variables do not exceed 11, *so we can choose* $M = 11$. *Thus* $K = 8.8$.

By using (3.9), *we can get* $\Delta_{max} \doteq 0.3$. *With* $\Delta = 0.2$, $x_0^T = (x_{10}, x_{20}, x_{30}, x40) = (1.1, 0.2, 2.8, 5.6)$ *and* $y_0^T = (y_{10}, y_{20}, y_{30}, y_{40}) = (2.3, 0.9, 0.5, 1.4)$, *a fourth order Runge-Kutta method with step size* $dt = 0.01$ *and evolution time* $T = 100$, *the simulation results are shown in Figure 7. We can see the original signal is recovered exactly.*

Example 5.4 *Chaotic masking communication by synchronizing two hysteresis systems with impulsive driving. The hysteresis system can be described as*

$$\begin{cases} \dot{x} = y + \gamma x + cz \\ \dot{y} = -\omega x - \delta_2 y \\ \epsilon\dot{z} = (1 - z^2)(Sx - D + z) - \delta_3 z, \end{cases} \tag{5.25}$$



(a) (b)

Figure 8: Simulation result by using the hysteresis systems with impulsive driving synchronization: (a) the recovered signal, (b) synchronization between x_1 and y_1

where γ, c, ω, δ_2, δ_3, ϵ, S, *and* D *are constants.*

If we take $(\gamma, c, \omega, \delta_2, \delta_3, \epsilon, S, D) = (0.2, 2.2, 10, 0.001, 0.001, 0.3, 1.667, 0.0)$, $x_0^T = (0.15264, -0.02281, 0.38127)$, $y_0^T = (0.75264, -0.4281, 0.58127)$, $\Delta = 0.4$, *a fourth order Runge-Kutta method with step size* $dt = 0.001$ *and evolution time* $T = 20$, *the simulation results are shown in Figure 8. Again, the original signal is recovered exactly.*

In Example 5.3 and Example 5.4, the frequancy of information signal is much lower than what we used in Example 5.1 and Example 5.2. The information signal can still be recovered exactly. Thus, the changing of signal frequencies can not destroy the synchronization by using impulsive driving. It should be noted that the estimation of the maximum time interval of two consecutive impulse instants is very conservative in the examples by using impulsive driving.

6 Conclusions

The simulation results shown that chaotic synchronization by impulsive driving can overcome the difficulty of information sensitivity by using self synchronization in the chaotic masking communication schemes. It may have some possible applications in secure communications if combinging with the traditional cryptographic technologies. In this paper, we only considered the ideal channel case. The real channel case should be explored further because it is very important to practical applications.

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