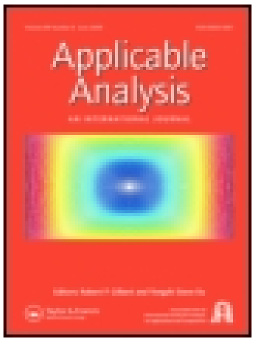

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Exponential Stability of Singularly Perturbed Systems with Time Delay

XINZHI LIU[a,*], XUEMIN SHEN[b] and YI ZHANG[a,**]

[a] *Department of Applied Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1;*
[b] *Department of Electrical and Computer Engineering, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1*

In this article, we study the exponential stability of singularly perturbed systems with time delay. By using vector delay inequalities and Lyapunov functions, exponential stability criteria are derived for both linear and some classes of nonlinear singularly perturbed systems with time delay. Examples are given to verify the stability criteria.

Keywords: Singular perturbation; Time delay; Exponential stability

AMS(MOS) Subject Classifications: 34D20; 34K20

1. INTRODUCTION

Singular perturbation techniques have been highly recognized and applied in a wide spectrum of fields such as mathematical modeling of physical systems, circuits, networks, fluid mechanics, etc. [1–8]. The stability properties of the singularly perturbed systems have been studied by many researchers. In [9], the stability bound of the singular perturbation parameter ϵ is found. A criterion in terms of the H_∞ norm is derived for a delay-dependent stability bound of the parameter ϵ [10]. Linear matrix inequality approach is proposed for the stability of singularly perturbed differential–difference systems [11]. Some results on asymptotical stability of singularly perturbed systems with small time delay are obtained in [12,13]. The previous studies have mainly focused on the linear and nonlinear singularly perturbed systems described by ordinary differential equations (ODEs), and the exponential stability issues have not been addressed. In this article, we study the exponential stability properties of the singularly perturbed systems with time delay. By establishing some vector delay inequalities and using Lyapunov functions, exponential stability criteria for linear and nonlinear singularly perturbed systems with time delay are derived. Examples are given to verify the criteria.

*Corresponding author. **Permanant address: Department of Mathematics Xuzhou Normal University, Xuzhou Jiangsu, China, 221009.

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The remainder of this article is organized as follows. Section 2 introduces some notations, definitions and lemmas. In Section 3, the exponential stability properties (criteria) for linear singularly perturbed systems with time delay are derived. Extension of the properties (criteria) to the nonlinear singularly perturbed systems with time delay are given in Section 4, followed by conclusions in Section 5.

2. PRELIMINARIES

To facilitate the discussion, it is convenient to introduce the following notations, definitions and lemmas.

Consider a singularly perturbed system with time delay

$$\begin{cases} x' = f(t, x, x(t-\tau), z, z(t-\tau)), \\ \epsilon z' = g(t, x, x(t-\tau), z), \end{cases} \tag{2.1}$$

where $x \in R^n$, $z \in R^m$ are slow and fast state variables, respectively, f and g are continuous for all variables, $f(t,0,0,0,0) \equiv g(t,0,0,0) \equiv 0$, τ is a constant time delay, and ϵ is a small positive parameter. For the system (2.1), the delayed fast variable is not included in the fast subsystem for simplicity.

Let $x_t = x(t+\theta), \theta \in [-\tau, 0], x(t, t_0, x_{t_0}, z_{t_0}), z(t, t_0, x_{t_0}, z_{t_0})$ be the solution of system (2.1) with initial functions $x(t_0, t_0, x_{t_0}, z_{t_0}) = x_{t_0}$, $z(t_0, t_0, x_{t_0}, z_{t_0}) = z_{t_0}$, $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ be the Euclidean norm of vector x, $\|A\| = \max_{1 \le j \le m}[\sum_{i=1}^{n} a_{ij}^2]^{1/2}$ be the norm of an $n \times m$ matrix $A = (a_{ij})$ and $\||x_t\|| = (\sum_{i=1}^{n} \sup_{\theta \in [t-\tau, t]} \|x_i(\theta)\|^2)^{1/2}$. Let A^T be the transpose of $n \times m$ matrix A, $\lambda(A)$ the eigenvalues of a $n \times n$ matrix A and $\operatorname{Re}\lambda(A)$ the real part of $\lambda(A)$. A symmetric matrix $P > 0$ is called positive definite, if all of its eigenvalues are positive. We write $x = (x_1, \ldots, x_n)^T \ge 0\ (\le 0)$ if $x_i \ge 0\ (\le 0)$ for all $i = 1, \ldots, n$.

Definition 2.1 System (2.1) is called exponentially stable if, there exist positive constants K and α such that

$$\|x(t)\| + \|z(t)\| \le K(\||x_{t_0}\|| + \||z_{t_0}\||)e^{-\alpha(t-t_0)}, \quad t \in [t_0, \infty),$$

for all solutions $x(t) = x(t, t_0, x_{t_0}, z_{t_0})$, $z(t) = z(t, t_0, x_{t_0}, z_{t_0})$.

Definition 2.2 A real $n \times n$ matrix Λ, with positive diagonal and nonpositive off-diagonal elements, is called on M-matrix if all its eigenvalues have positive real part.

LEMMA 2.1 *Let $A(t)$ be an $n \times n$ matrix of continuous functions defined on the interval $J = [0, +\infty)$ and*

(i) $\operatorname{Re}\lambda(A(t)) \le -c_1 < 0, \quad \forall t \in J;$
(ii) $\|A(t)\| \le c_2, \quad \|A'(t)\| \le c_2, \quad \forall t \in J;$

then there exists a positive definitive matrix $P(t)$ such that

$$A^T(t)P(t) + P(t)A(t) = -I,$$

where c_1, c_2 are constants and I is the identity matrix.

The proof of Lemma 2.1 may be found in [4] and the next lemma is taken from [9].

LEMMA 2.2 *Let $\alpha, \beta, u \in C(R, R^+)$, $\alpha(t) - \beta(t) > \lambda^* > 0$ $\beta(t)$ is bounded and*

$$u'(t) \le -\alpha(t)u(t) + \beta(t) \sup_{\vartheta \in [t-\tau,\, t]} u(\vartheta),$$

then there exists $\lambda > 0$, such that

$$u(t) \le \sup_{\theta \in [t_0-\tau,\, t_0]} u(\vartheta) e^{-\lambda(t-t_0)}, \quad t \ge t_0$$

The following lemma will be used frequently in the next two sections.

LEMMA 2.3 *Let $A(t)$ and $B(t)$ be defined the same as in Lemma 2.1. Assume that*

(i) $\lambda(A(t) + A^T(t)) < -\alpha(t) < 0$, *$B(t)$ is bounded*;
(ii) $-\alpha(t) + 2\|B(t)\| \le -\beta < 0$, *$\beta$ is a constant*;
(iii) $y'(t) \le A(t)y(t) + B(t)\sup_{t-\tau \le \theta \le t} y(\theta)$,

where $y = (y_1, y_2)^T \ge 0$; $\sup_{t-\tau\le\theta\le t} y(\theta) = (\sup_{t-\tau\le\theta\le t} y_1(\theta), \sup_{t-\tau\le\theta\le t} y_2(\theta))^T$. Then there exists $\alpha^* > 0$, such that

$$\|y\| \le \| |y_{t_0}| \| e^{-\alpha^*(t-t_0)}, \quad t \in [t_0 - \tau, \infty). \tag{2.2}$$

Proof Let $v(t) = \|y\|^2 = y^T y$, then

$$\begin{aligned} \frac{dv(t)}{dt} &\le y^T(t)(A^T(t) + A(t))y(t) + 2y^T(t)B(t) \sup_{t-\tau\le\theta\le t} y(\theta) \\ &\le -\alpha(t)\|y(t)\|^2 + 2\|B(t)\| \|y(t)\| \| |y_t| \| \\ &\le -\alpha(t)\|y(t)\|^2 + \|B(t)\|(\|y(t)\|^2 + \| |y_t| \|^2) \\ &\le [-\alpha(t) + \|B(t)\|]\|y(t)\|^2 + \|B(t)\| \| |y_t| \|^2 \\ &\le [-\alpha(t) + \|B(t)\|]v(t) + \|B(t)\| \| |v_t| \|. \end{aligned}$$

By condition (ii) and Lemma 2.2, we get (2.2). The proof is complete.

3. LINEAR SYSTEMS

In this Section, we shall study the exponential stability of the following linear singularly perturbed system with time delay defined by (2.1).

$$\begin{cases} x' = A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t, \\ \epsilon z' = A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)z, \end{cases} \tag{3.3}$$

where $x \in R^n$, $z \in R^m$, $x_t \in C([t-\tau, t], R^n)$, $z_t \in C([t-\tau, t], R^m)$, $A_{ij}(t)$, $B_{ij}(t)$ are matrices with appropriate dimensions, $1 \le i$, $j \le 2$, $\tau \ge 0$ is a constant and $A_{11}(t)$,

$A_{21}(t)$, $A_{22}(t)$, $B_{21}(t)$ are continuously differentiable. $B_{21}(t)$ is assumed to be nonsingular for every t.

(S_1): Assume that there exist constants γ and μ, such that

$$\operatorname{Re}\lambda(B_{21}(t)) \le -\gamma, \quad \|B_{21}(t)\| \le \mu, \quad \|B'_{21}(t)\| \le \mu;$$

$$\operatorname{Re}\lambda(A_{11}(t)) \le -\gamma, \quad \|A_{11}(t)\| \le \mu, \quad \|A'_{11}(t)\| \le \mu;$$

$$\|B_{21}^{-1}(t)A_{21}(t)\| \le \mu, \quad \|B_{21}^{-1}(t)A_{22}(t)\| \le \mu,$$

for all $t \in R^+$.

From (S_1) and Lemma 2.1, there exist differentiable positive matrices $P_1(t)$ and $P_2(t)$, such that

$$A_{11}^T(t)P_1(t) + P_1(t)A_{11}(t) = -I_n, \tag{3.4}$$

$$B_{21}^T(t)P_2(t) + P_2(t)B_{21}(t) = -I_m, \tag{3.5}$$

where I_n, I_m are $n \times n$, $m \times m$ identity matrices, respectively. The solutions of (3.4) and (3.5) are given by

$$P_1(t) = \int_0^\infty e^{A_{11}^T(t)\sigma} e^{A_{11}(t)\sigma} d\sigma,$$

and

$$P_2(t) = \int_0^\infty e^{B_{21}^T(t)\sigma} e^{B_{21}(t)\sigma} d\sigma,$$

respectively. There exist positive constants M_1, M_2 and α_i, β_i $(i = 1, 2)$, such that

$$M_1 \le \|P_1(t)\| \le M_2; \quad M_1 \le \|P_2(t)\| \le M_2;$$
$$\alpha_1\|x\|^2 \le x^T P_1(t)x \le \beta_1\|x\|^2;$$
$$\alpha_2\|z\|^2 \le z^T P_2(t)z \le \beta_2\|z\|^2.$$

THEOREM 3.1 *If (S_1) holds, and*

(1) *there exist functions $a_{ij}(t)$, $b_{ij}(t)$, $i, j = 1, 2$, satisfying*

$$2x^T P_1(A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t) + x^T P'_1(t)x \le a_{11}(t)\|x\|^2 + a_{12}(t)\||x_t|\|^2 + b_{11}(t)\|z-h\|^2 + b_{12}(t)\||(z-h)_t|\|^2,$$

and

$$
\begin{aligned}
&- 2(z-h)^T P_2 h' + (z-h)^T P_2'(z-h) \\
&\le a_{21}(t)\|x\|^2 + a_{22}(t)\||\bar{x}_t|\|^2 + b_{21}(t)\|z-h\|^2 + b_{22}\||(z-h)_t|\|^2,
\end{aligned}
$$

where $h = -B_{21}^{-1}(t)[A_{21}(t)x(t) + A_{22}(t)x_t]$, *and* $a_{i2}(t)$, $b_{i2}(t)$ *are bounded;*

(2) *there exist positive numbers* ϵ^* *and* η, *such that* $-\widetilde{A(t)}$ *is M-matrix and* $\lambda(\tilde{A} + A) + 2\|\tilde{B}\| \le \eta$ *for all* $t \in R^+$, *where*

$$
\widetilde{A(t)} = \begin{pmatrix} -\dfrac{1-a_{11}(t)}{\beta_1} & \dfrac{b_{11}(t)}{\alpha_2} \\[2ex] \dfrac{a_{21}(t)}{\alpha_1} & -\dfrac{1-\epsilon^* b_2(t)}{\epsilon^* \beta_2(t)} \end{pmatrix}, \qquad \widetilde{B(t)} = \begin{pmatrix} \dfrac{a_{12}(t)}{\alpha_1} & \dfrac{b_{12}(t)}{\alpha_2} \\[2ex] \dfrac{a_{22}(t)}{\alpha_1} & \dfrac{b_{22}(t)}{\alpha_2} \end{pmatrix},
$$

then the singularly perturbed time delay system (3.3) *is exponentially asymptotically stable for all* $\epsilon \in (0, \epsilon^*]$.

Proof Let $W(t,x,z) = (z-h)^T P_2(t)(z-h)$, $V(t,x) = x^T P_1(t)x$, then the derivative of W along with system (3.3) is given by

$$
\begin{aligned}
W' &= (z'-h')^T P_2(t)(z-h) + (z-h)^T P_2(t)(z'-h') \\
&\quad + (z-h)^T P_2'(t)(z-h) \\
&= \left[\frac{1}{\epsilon}\big(A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)z\big) - h'\right]^T P_2(t)(z-h) \\
&\quad + (z-h)^T P_2(t)\left[\frac{1}{\epsilon}(A_{21}(t)x + A_{22}(t)x_t + B_{21}(t)z) - h'\right] \\
&\quad + (z-h)^T P_2'(t)(z-h) \\
&= \frac{1}{\epsilon}(z-h)^T\big[B_{21}^T(t)P_2(t) + P_2(t)B_{21}(t)\big](z-h) \\
&\quad - 2(z-h)^T P_2(t)h' + (z-h)^T P_2'(t)(z-h) \\
&\le -\frac{1}{\epsilon}(1-\epsilon b_{21}(t))\|z-h\|^2 + a_{21}(t)\|x\|^2 \\
&\quad + a_{22}(t)\||x_t|\|^2 + b_{22}(t)\||(z-h)_t|\|^2,
\end{aligned}
$$

and the derivative of $V = x^T P_1(t)x$ along with system (3.3) is given by

$$\begin{aligned} V' &= [A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t]^T P_1(t)x \\ &\quad + x^T P_1(t)[A_{11}(t)x + A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t] \\ &\quad + x^T P_1'(t)x \\ &\le x^T(A_{11}^T(t)P_1(t) + P_1(t)A_{11}(t))x \\ &\quad + 2x^T P_1(t)(A_{12}(t)x_t + B_{11}(t)z + B_{12}(t)z_t) \\ &\quad + x^T P_1'(t)x \\ &\le -(1 - a_{11}(t))\|x\|^2 + a_{12}(t)\||x_t|\|^2 \\ &\quad + b_{11}(t)\|z - h\|^2 + b_{12}(t)\||(z-h)_t|\|^2. \end{aligned}$$

The property of $-A^*(t)$ being M-matrix implies $((1 - \epsilon^* b_{21}(t))/\epsilon^* \beta_2(t)) > 0$. Since $-1/\epsilon$ is an increasing function of ϵ, then

$$V' \le -\frac{1 - a_{11}(t)}{\beta_1} V + \frac{b_{11}(t)}{\alpha_2} W + \frac{a_{12}(t)}{\alpha_1} \||V_t|\| + \frac{b_{12}(t)}{\alpha_2} \||W_t|\|$$

and

$$\begin{aligned} W' &\le \frac{a_{21}(t)}{\alpha_1} V - \frac{1 - \epsilon b_{21}(t)}{\epsilon \beta_2} W + \frac{a_{22}(t)}{\alpha_1} \||V_t|\| + \frac{b_{22}(t)}{\alpha_2} \||W_t|\| \\ &\le \frac{a_{21}(t)}{\alpha_1} V - \frac{1 - \epsilon^* b_{21}(t)}{\epsilon^* \beta_2} W + \frac{a_{22}(t)}{\alpha_1} \||V_t|\| + \frac{b_{22}(t)}{\alpha_2} \||W_t|\|, \quad \epsilon \in (0, \epsilon^*]. \end{aligned}$$

By Lemma 2.3, it follows that there exists a positive constant $\lambda > 0$, such that

$$V \le (\||V_{t_0}|\| + \||W_{t_0}|\|)e^{-2\lambda(t-t_0)},$$

and

$$W \le (\||V_{t_0}|\| + \||W_{t_0}|\|)e^{-2\lambda(t-t_0)}.$$

Thus, there exists a positive constant K_1, such that

$$\|x\| \le K_1(\||x_{t_0}|\| + \||z_{t_0}|\|)e^{-\lambda(t-t_0)} \tag{3.6}$$

because $\||(z-h)_{t_0}|\| \le \||z_{t_0}|\| + \||h_{t_0}|\|$. Moreover, (3.6) and

$$\|z\| - \|h\| \le \|(z-h)(t)\| \le (\alpha_2)^{-1/2} W^{1/2}$$

imply that there exists a constant $K_2 > 0$ such that

$$\|z\| \le K_2(\||x_{t_0}|\| + \||z_{t_0}|\|)e^{-\lambda(t-t_0)},$$

where $h = -B_{21}^{-1}(t)[A_{21}(t)x(t) + A_{22}(t)x(t-\tau)]$. The proof is complete.

THEOREM 3.2 *Consider the Case* 1 *of the system* (3.3) *with constant coefficient*

$$\begin{cases} x' = A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau), \\ \epsilon z' = A_{21}x + B_{21}z. \end{cases} \tag{3.7}$$

Assume that

(1) $\lambda(A_{11}^T + A_{11}) \le -\lambda_1 < 0$; $\lambda(B_{21}^T + B_{21}) \le -\lambda_2 < 0$;

(2) $-\lambda_1 + a_{12} + b_{12} + \|\tilde{B}\| < 0$, *where* $\tilde{B} = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix}$,

$$\|A_{12} - B_{12}B_{21}^{-1}A_{21}\| = a_{12}, \quad \|B_{12}\| = b_{12}, \quad \|B_{21}^{-1}A_{21}A_{11}\| = a_{21},$$
$$\|B_{21}^{-1}A_{21}A_{12}\| = a_{22}, \quad \|B_{21}^{-1}A_{21}B_{12}\| = b_{22}.$$

then there exists ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$ *system* (3.7) *is exponentially stable.*

Proof We shall prove that the positive constant ϵ^* is required to satisfy $-\lambda_2/\epsilon^* + a_{21} + a_{22} + b_{22} + \|\tilde{B}\| \le -\lambda_1 + a_{12} + b_{12} + \|\tilde{B}\| < 0$.

Let $h = -B_{21}^{-1}A_{21}x$, $V = x^T x$ and $W = (z-h)^T(z-h)$. Then the derivative of V along with the system (3.7) is

$$\begin{aligned} V' &= (x^T)'x + x^T x' \\ &= [A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau)]^T x \\ &\quad + x^T[A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau)] \\ &= x^T[A_{11}^T + A_{11}]x + 2x^T[A_{12}x(t-\tau) + B_{12}z(t-\tau)] \\ &\le -\lambda_1 x^T x + 2x^T[A_{12}x(t-\tau) + B_{12}h(t-\tau) + B_{12}(z(t-\tau) - h(t-\tau))] \\ &\le -\lambda_1 x^T x + 2x^T[(A_{12} - B_{12}B_{21}^{-1}A_{21})x(t-\tau) + B_{12}(z(t-\tau) - h(t-\tau))] \\ &\le (-\lambda_1 + a_{12} + b_{12})V + a_{12}|||V_t||| + b_{12}|||W_t|||. \end{aligned}$$

The derivative of W along with (3.7) is

$$\begin{aligned} W' &= [(z-h)^T]'(z-h) + (z-h)^T(z-h)' \\ &= \left\{\frac{1}{\epsilon}(A_{21}x + B_{21}z) + B_{21}^{-1}A_{21}[A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau)]\right\}^T \\ &\quad \times (z-h) + (z-h)^T\left\{\frac{1}{\epsilon}(A_{21}x + B_{21}z) + B_{21}^{-1}A_{21}[A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau)]\right\} \\ &\le -\frac{\lambda_2}{\epsilon}(z-h)^T(z-h) + 2(z-h)^T B_{21}^{-1}A_{21}[A_{11}x + A_{12}x(t-\tau) + B_{12}z(t-\tau)] \\ &\le -\frac{\lambda_2}{\epsilon}\|z-h\|^2 + 2\|(z-h)\|\,[a_{21}\|x\| + a_{22}|||x_t||| + b_{22}|||w_t|||] \\ &\le \left[-\frac{\lambda_2}{\epsilon} + a_{21} + a_{22} + b_{22}\right]W + a_{21}V + a_{22}|||V_t||| + b_{22}|||W_t|||. \end{aligned}$$

In view of (2) and Lemma 2.3, for any $\epsilon \in (0, \epsilon^*)$, there exists $\alpha > 0$ such that

$$\|x\|^2 = V \leq [\||V_{t_0}|\| + \||W_{t_0}|\|]e^{-2\alpha(t-t_0)}$$

$$\|z - h\|^2 = W \leq [\||V_{t_0}|\| + \||W_{t_0}|\|]e^{-2\alpha(t-t_0)}.$$

Similar to the proof of Theorem 3.1, it follows that there exists $K > 0$, such that

$$\|x\| \leq K[\||x_{t_0}|\| + \||z_{t_0}|\|]e^{-\alpha(t-t_0)}$$

$$\|z\| \leq K[\||x_{t_0}|\| + \||z_{t_0}|\|]e^{-\alpha(t-t_0)}.$$

The proof is complete.

Example 3.1 Consider the system

$$\begin{cases} x' = -2x + z(t-\tau), \\ \epsilon z' = 3x - 3z. \end{cases} \tag{3.8}$$

Using the notations of Theorem 3.2, we have

(1) $\lambda_1 = 4$; $\lambda_2 = 6$; $a_{12} = 1$; $a_{21} = 2$; $a_{22} = 0$; $b_{12} = 1$; $b_{22} = 1$; $B^* = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and
(2) $-\lambda_1 + a_{12} + b_{12} + \|\tilde{B}\| < 0$,

By Theorem 3.2, there exists $\epsilon^* = 1.2$ such that for any $\epsilon \in (0, \epsilon^*]$ system (3.8) is exponentially stable.

THEOREM 3.3 *Consider the Case* 2 *of the system* (3.3) *with constant coefficient*

$$\begin{cases} x' = A_{11}x + B_{11}z(t), \\ \epsilon z' = A_{22}x(t-\tau) + B_{21}z. \end{cases} \tag{3.9}$$

Assume that

(1) $\lambda(A_{11}^T + A_{11}) \leq -\lambda_1 < 0$; $\lambda(B_{21}^T + B_{21}) \leq -\lambda_2 < 0$;

(2) $-\lambda_1 + a_{12} + b_{11} + \|\tilde{B}\| < 0$, *where* $\tilde{B} = \begin{pmatrix} a_{12} & 0 \\ a_{22} & b_{22} \end{pmatrix}$,

$$\|B_{11}B_{21}^{-1}A_{22}\| = a_{12}, \quad \|B_{11}\| = b_{11}, \quad \|B_{21}^{-1}A_{22}A_{11}\| = a_{22},$$

$$\|B_{21}^{-1}A_{22}B_{11}B_{21}^{-1}A_{22}\| = a_{22}^*, \quad \|B_{21}^{1}A_{22}B_{11}\| = b_{22}.$$

then there exists ϵ^* *such that for any* $\epsilon \in (0, \epsilon^*]$ *system* (3.9) *is exponentially stable.*

Proof We shall prove that the positive number ϵ^* is required to satisfy $-\lambda_2/\epsilon^* + a_{22} + b_{22} + \|B^*\| \leq -\lambda_1 + a_{12} + b_{11} + \|B^*\| < 0$ is required.

Let $h = -B_{21}^{-1}A_{22}x(t-\tau)$, $V = x^T x$ and $W = (z-h)^T(z-h)$. Then the derivative of V along with the system (3.9) is

$$\begin{aligned} V' &= (x^T)'x + x^T x' = [A_{11}x + B_{11}z(t)]^T x + x^T[A_{11}x + B_{11}z(t)] \\ &= x^T[A_{11}^T + A_{11}]x + 2x^T[B_{11}(z-h) + B_{11}B_{21}^{-1}A_{22}x(t-\tau)] \\ &\le (-\lambda_1 + a_{12} + b_{11})V + a_{12}V(t-\tau) + b_{11}W. \end{aligned}$$

The derivative of W along (3.9) is

$$\begin{aligned} W' &= [(z-h)^T]'(z-h) + (z-h)^T(z-h)' \\ &= \left\{\frac{1}{\epsilon}(A_{22}x(t-\tau) + B_{21}z) + B_{21}^{-1}A_{22}[A_{11}x(t-\tau) + B_{11}z(t-\tau)]\right\}^T (z-h) \\ &\quad \times (z-h)^T\left[\frac{1}{\epsilon}(A_{22}x(t-\tau) + B_{21}z) + B_{21}^{-1}A_{22}[A_{11}x(t-\tau) + B_{11}z(t-\tau)\right] \\ &\le -\frac{1}{\epsilon}(z-h)^T(B_{21}^T + B_{21})(z-h) + 2(z-h)^T B_{21}^{-1}A_{22} \\ &\quad \times [A_{11}x(t-\tau) + B_{11}(z(t-\tau) - h(t-\tau)) + B_{11}B_{21}^{-1}A_{22}x(t-2\tau)] \\ &\le -\frac{\lambda_2}{\epsilon}\|z-h\|^2 + (a_{22} + a_{22}^* + b_{22})\|z-h\|^2 + a_{22}\||x(t-\tau)|\|^2 \\ &\quad + b_{22}\||(z-h)(t-\tau)|\|^2 + a_{22}^*\||x(t-2\tau)|\|^2 \\ &\le \left[-\frac{\lambda_2}{\epsilon} + a_{22} + a_{22}^* + b_{22}\right]W + (a_{22} + a_{22}^*)\||V_t|\| + b_{22}\||W_t|\|. \end{aligned}$$

In view of (2) and Lemma 2.3, for any $\epsilon \in (0, \epsilon^*)$, there exists $\alpha > 0$ such that

$$\|x\|^2 = V \le [\||V_{t_0}|\| + \||W_{t_0}|\|]e^{-2\alpha(t-t_0)},$$

$$\|z-h\|^2 = W \le [\||V_{t_0}|\| + \||W_{t_0}|\|]e^{-2\alpha(t-t_0)}.$$

Similar to the proof of Theorem 3.1, it follows that there exists $K > 0$, such that

$$\|x\| \le K[\||x_{t_0}|\| + \||z_{t_0}|\|]e^{-\alpha(t-t_0)},$$

$$\|z\| \le K[\||x_{t_0}|\| + \||z_{t_0}|\|]e^{-\alpha(t-t_0)}.$$

The proof is complete.

Example 3.2 Consider the system

$$\begin{cases} x' = -2x + z(t), \\ \epsilon z' = 3x(t-\tau) - 6z. \end{cases} \tag{3.10}$$

Using the notations of Theorem 3.3, we have

(1) $\lambda_1 = 4$; $\lambda_2 = 6$; $a_{12} = 1/2$; $a_{22} = 1$; $a^*_{22} = 1/4$; $b_{22} = 1/2$; $b_{11} = 1$; $\begin{pmatrix} 1/2 & 0 \\ 1 & 1/2 \end{pmatrix}$
(2) $-\lambda_1 + a_{12} + b_{11} + \|\tilde{B}\| < 0$.

By Theorem 3.3, set $\epsilon^* = 3$ then for any $\epsilon \in (0, \epsilon^*]$ system (3.10) is exponentially stable.

The study of Cases 1 and 2 demonstrate the obtained stability criteria of Theorem 3.1. Other cases of system (3.7) can be studied similarly.

4. NONLINEAR SYSTEMS

Consider a nonlinear autonomous singularly perturbed system

$$\begin{cases} x' = f(x, x_t, z, z_t), \ \ x \in R^n, \ \ z \in R^m, \\ \epsilon z' = B_{21} z + B(x, x_t), \end{cases} \tag{4.11}$$

where $f(x, x_t, z, z_t) = A_{11}x + g(x, x_t, z, z_t)$ and $f(0,0,0,0) \equiv 0$, $B(0,0) \equiv 0$, $x_t \in C([t-\tau, t], R^n)$, $0 \le \tau < \infty$. Assume that (4.11) has a unique equilibrium at the origin $x = (0, \ldots, 0)^T$, $z = (0, \ldots, 0)^T$ and f, B are smooth enough to ensure that, for continuous initial functions, (4.11) has a unique solutions.

(S_2): Assume that there exists positive constant c such that

$$\operatorname{Re} \lambda(A_{11}) \le -c; \quad \operatorname{Re} \lambda(B_{21}) \le -c,$$

If (S_2) holds, then there exist positive matrices P_1 and P_2, such that

$$B_{21}^T P_2 + P_2 B_{21} = -I_m,$$

$$A_{11}^T P_1 + P_1 A_{11} = -I_n,$$

where I_m, I_n are the identity matrices, and positive constants α_i, β_i ($i = 1, 2$) such that

$$\alpha_i \|x\|^2 \le x^T P_i x \le \beta_i \|x\|^2; \quad i = 1, 2.$$

THEOREM 4.1 *Suppose that* (S_2) *holds and*

(1) *there exist constants* a_{ij}, b_{ij}, $i, j = 1, 2$, *such that*

$$-2(z-h)^T P_2 h' \le a_{21}\|x\|^2 + a_{22}\||x_t|\|^2 + b_{21}\|z-h\|^2 + b_{22}\||(z-h)_t|\|^2,$$

$$2x^T P_1 g(x, x_t, z, z_t) \le a_{11}\|x\|^2 + a_{12}\||x_t|\|^2 + b_{11}\|z-h\|^2 + b_{21}\||(z-h)_t|\|^2,$$

where $h = -A^{-1}(x)B(x, x_t)$;

(2) *there exist* ϵ^* *and* η *such that* $-\tilde{A}$ *is a M-matrix and* $\lambda((\tilde{A})^T + \tilde{A}) + 2\|\tilde{B}\| \le -\eta$ *where*

$$\tilde{A} = \begin{pmatrix} -\dfrac{1-a_{11}}{\beta_1} & \dfrac{b_{11}}{\alpha_1} \\ \dfrac{a_{21}}{\alpha_2} & -\dfrac{1-\epsilon^* b_{21}}{\epsilon^*\beta_2} \end{pmatrix}, \qquad \widetilde{B(t)} = \begin{pmatrix} \dfrac{a_{12}}{\alpha_1} & \dfrac{b_{12}}{\alpha_2} \\ \dfrac{a_{22}}{\alpha_1} & \dfrac{b_{22}}{\alpha_2} \end{pmatrix},$$

then the nonlinear singularly perturbed time delay system (4.11) *is exponentially asymptotically stable for all* $\epsilon \in (0, \epsilon^*)$.

Proof Let $W = (z-h)^T P_2(z-h)$, $V = x^T P_1 x$. The derivative of W along with system (4.11) is given by

$$\begin{aligned} W' &= (z'-h')^T P_2(z-h) + (z-h)^T P_2(z'-h') \\ &= \left[\frac{1}{\epsilon}(B_{21}z + B(x,x_t)) - h'\right]^T P_2(z-h) \\ &\quad + (z-h)^T P_2\left[\frac{1}{\epsilon}(B_{21}z + B(x,x_t)) - h'\right] \\ &= \frac{1}{\epsilon}(z-h)^T[B_{21}^T P_2 + P_2 B_{21}](z-h) - 2(z-h)^T P_2 h' \\ &\le -\frac{1}{\epsilon}(1-\epsilon b_{21})\|z-h\|^2 + a_{21}\|x\|^2 \\ &\quad + a_{22}\||x_t|\|^2 + b_{22}\||(z-h)_t|\|^2. \end{aligned}$$

The derivative of $V = x^T P_1 x$ along with system (4.11) is given by

$$\begin{aligned} V' &= (A_{11}x + g(x,x_t,z,z_t))^T P_1 x + x^T P_1(A_{11}x + g(x,x_t,z,z_t)) \\ &\le x^T(A_{11}^T P_1 + P_1 A_{11})x + 2x^T P_1 g(x,x_t,z,z_t) \\ &\le -(1-a_{11})\|x\|^2 + a_{12}\||x_t|\|^2 + b_{11}\|z-h\|^2 + b_{12}\||(z-h)_t|\|^2. \end{aligned}$$

Similar to the proof of Theorem 3.1, there exists a positive constant $\lambda > 0$, such that for any $\epsilon \in (0, \epsilon^*)$,

$$\begin{aligned} V &\le (\||V_{t_0}|\| + \||W_{t_0}|\|)e^{-\lambda(t-t_0)}, \\ W &\le (\||V_{t_0}|\| + \||W_{t_0}|\|)e^{-\lambda(t-t_0)}, \end{aligned}$$

and there exist positive constants K_1, K_2 such that

$$\begin{aligned} \|x\| &\le K_1(\||x_{t_0}|\| + \||z_{t_0}|\|)e^{-\lambda(t-t_0)/2}, \\ \|z\| &\le K_2(\||x_{t_0}|\| + \||z_{t_0}|\|)e^{-\lambda(t-t_0)/2}. \end{aligned}$$

The proof is complete.

THEOREM 4.2 *Consider a special Case of system* (4.11) *with constant coefficient*

$$\begin{cases} x' = A_{11}x + A_{12}(x(t-\tau)) + B_{11}(z), \quad x \in R^n,\ z \in R^m, \\ \epsilon z' = A_{22}x(t-\tau) + B_{21}z, \end{cases} \tag{4.12}$$

where A_{11}, A_{22} *and* B_{21} *are constant matrices*, A_{12}, *and* B_{11} *are continuous functional vectors.*

Assume that

(1) $\lambda(A_{11}^T + A_{11}) \le \lambda_1 < 0$, $\lambda(B_{21}^T + B_{21}) \le \lambda_2 < 0$;
(2) *there exist* $a_{ij} \ge 0$, $b_{ij} \ge 0$, $i,j = 1,2$, *such that*

$$2x^T(A_{12}(x(t-\tau)) + B_{11}(z)) \le a_{11}\|x\|^2 + a_{12}\||x_t|\|^2 + b_{11}\|z-h\|^2 + b_{12}\||(z-h)_t|\|^2,$$

$$\begin{aligned} &2(z-h)^T B_{21}^{-1} A_{22}(A_{11}x(t-\tau) + A_{12}(x(t-2\tau)) + B_{11}(z(t-\tau))) \\ &\quad \le a_{22}\||x_t|\|^2 + b_{21}\|z-h\|^2 + b_{22}\||(z-h)_t|\|^2, \end{aligned}$$

where $h = B_{21}^{-1}A_{22}x(t-\tau)$;
(3) $\lambda(\tilde{A} + \tilde{A}^T) + 2\|\tilde{B}\| < 0$, *where*

$$\tilde{A} = \begin{pmatrix} -\lambda_1 + a_{11} & b_{11} \\ a_{21} & -\dfrac{\lambda_2}{\epsilon} + b_{21} \end{pmatrix}, \qquad \widetilde{B(t)} = \begin{pmatrix} a_{12} & b_{12} \\ a_{22} & b_{22} \end{pmatrix},$$

then there exists a $\epsilon^* > 0$ *such that the singularly perturbed system* (4.12) *is exponentially asymptotically stable for all* $\epsilon \in (0, \epsilon^*]$.

Proof Let $W(t,x,z) = (z-h)^T(z-h)$, $V(t,x) = x^T x$. The derivative of W along with system (4.12) is given by

$$\begin{aligned} W' &= (z' - h')^T(z-h) + (z-h)^T(z'-h') \\ &= \bigg[\frac{1}{\epsilon}(A_{22}x(t-\tau) + B_{21}z) + B_{21}^{-1}A_{22}(A_{11}x(t-\tau) + A_{12}(x(t-2\tau)) \\ &\qquad + B_{11}(z(t-\tau)))\bigg]^T (z-h) + (z-h)^T\bigg[\frac{1}{\epsilon}(A_{22}x(t-\tau) + B_{21}z) \\ &\qquad + B_{21}^{-1}A_{22}(A_{11}x(t-\tau) + A_{12}(x(t-2\tau)) + B_{11}(z(t-\tau)))\bigg] \\ &\le \frac{1}{\epsilon}(z-h)^T(B_{21}^T + B_{21})(z-h) \\ &\quad + a_{22}\||x_t|\|^2 + b_{21}\|z-h\|^2 + b_{22}\||(z-h)_t|\|^2 \\ &\le \left(-\frac{1}{\epsilon}\lambda_2 + b_{21}\right)\|z-h\|^2 + a_{22}\||x_t|\|^2 + b_{22}\||(z-h)_t|\|^2 \\ &\le \left(-\frac{1}{\epsilon}\lambda_2 + b_{21}\right)W + a_{22}\||V_t|\| + b_{22}\||W_t|\|; \end{aligned}$$

the derivative of $V = x^T P_1 x$ along with system (4.12) is given by

$$\begin{aligned}
V' &= (A_{11}x + A_{12}(x(t-\tau)) + B_{11}(z))^T x + x^T(A_{11}x + A_{12}(x(t-\tau)) + B_{11}(z) \\
&= x^T(A_{11}^T + A_{11})x + 2x^T(A_{12}(x(t-\tau)) + B_{11}^T(z)) \\
&\le -\lambda_1\|x\|^2 + a_{11}\|x\|^2 + a_{12}|||x_t|||^2 + b_{11}\|z-h\|^2 + b_{12}|||(z-h)_t|||^2 \\
&\le (-\lambda_1 + a_{11})V + a_{12}|||V_t||| + b_{11}W + b_{12}|||W_t|||.
\end{aligned}$$

Similar to the proof of Theorem 3.1, there exists a positive constant $\lambda > 0$, such that for any $\epsilon \in (0, \epsilon^*)$,

$$V \le (|||V_{t_0}||| + |||W_{t_0}|||)e^{-\lambda(t-t_0)},$$
$$W \le (|||V_{t_0}||| + |||W_{t_0}|||)e^{-\lambda(t-t_0)},$$

and there exist positive constants K_1, K_2 such that

$$\|x\| \le K_1(|||x_{t_0}||| + |||z_{t_0}|||)e^{-\lambda(t-t_0)/2},$$
$$\|z\| \le K_2(|||x_{t_0}||| + |||z_{t_0}|||)e^{-\lambda(t-t_0)/2}.$$

The proof is complete.

Example 4.1 Consider a nonlinear singularly perturbed time delay system

$$\begin{cases} x' = -23x + 9ln(1 + x^2(t-\tau)) + \sin z, \quad x \in R^1,\ z \in R^1, \\ \epsilon z' = x(t-\tau) - 4z, \end{cases} \tag{4.13}$$

Let $h = (1/4)x(t-\tau)$, $V = x^2$, $W = (z-h)^2$, then

$$\begin{aligned}
V' &= 2x[-23x + 9ln(1 + x^2(t-\tau)) + \sin z] \\
&\le -46x^2 + 18|xx(t-\tau)| + 2[x(z-h) + xh] \\
&\le -46x^2 + 9x^2 + 9x(t-\tau)^2 + x^2 + (z-h)^2 + x^2 + \frac{1}{16}x(t-\tau)^2 \\
&\le -35V + W + 10|||V_t|||; \\
W' &= 2(z-h)\left[\frac{1}{\epsilon}(x(t-\tau) - 4z) - \frac{1}{4}(-23x(t-\tau)\right. \\
&\qquad \left. + 9ln(1 + x^2(t-2\tau)) + \sin z(t-\tau))\right] \\
&\le -\frac{8}{\epsilon}(z-h)^2 + \frac{1}{2}|z-h|\big[23|x(t-\tau)| \\
&\quad + 9|x(t-2\tau)| + |z(t-\tau) - h(t-\tau)| + |h(t-\tau)|\big] \\
&\le -\frac{8}{\epsilon}W + \frac{23}{4}W + \frac{23}{4}|||V_t||| \\
&\quad + \frac{9}{4}W + \frac{9}{4}|||V_t||| + \frac{1}{4}W + \frac{1}{4}|||W_t||| + \frac{1}{4}W + \frac{1}{64}|||V_t||| \\
&\le \left(-\frac{8}{\epsilon} + 9\right)W + 9|||V_t||| + |||W_t|||.
\end{aligned}$$

Using the notations of Theorem 4.2,

$$\tilde{A} = \begin{pmatrix} -35 & 1 \\ 0 & -\frac{8}{\epsilon} + 9 \end{pmatrix}, \qquad \widetilde{B(t)} = \begin{pmatrix} 10 & 0 \\ 9 & 1 \end{pmatrix},$$

then $\lambda(\tilde{A} + \tilde{A}^T) + 2\|\tilde{B}\| < 0$. By Theorem 4.2, the singularly perturbed system (4.13) is exponentially asymptotically stable for all $\epsilon \in (0, \epsilon^*]$, where $\epsilon^* = 1/3$.

5. CONCLUSIONS

Exponential stability criteria of singularly perturbed systems with time delay are obtained. Although single delay has been considered only, the study of the exponential stability criteria can be extended to the case with multiple delays.

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