

# Stability of switched systems with time delay

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## Abstract

In this paper, we study the qualitative properties of linear and nonlinear delay switched systems which have stable and unstable subsystems. First, we prove some inequalities which lead to the switching laws that guarantee: (a) the global exponential stability to linear switched delay systems with stable and unstable subsystems; (b) the local exponential stability of nonlinear switched delay systems with stable and unstable subsystems. In addition, these switching laws indicate that if the total activation time ratio among the stable subsystems, unstable subsystems and time delay is larger than a certain number, the switched systems are exponentially stable for any switching signals under these laws. Some examples are given to illustrate the main results.

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## 1. Introduction

Switched systems are systems that consist of several subsystems and controlled by switching laws. Such systems are often encountered in biochemical systems, control systems, etc., (see [5,4]). For switched systems, one of the most important and challenging problems is to find the switching laws, i.e., what switching laws can guarantee the switched systems stable.

Recently, there has been increasing interest in the stability analysis of switched systems, and switching control design of such systems (see [8,9,11–13] and the references therein). Using common Lyapunov functions, [18] studies stability for linear switched systems and shows that a common Lyapunov function exists when the stability matrices  $A_i$  commute pairwise, i.e.  $A_i A_j = A_j A_i$ ,  $i, j = 1, 2, \dots, N$ . Exponential stability is studied for some special linear time-invariant switched system in [23,11]. Similar results are presented in [7]. Lie algebra is used to prove the existence of common Lyapunov function and then the stability of switched systems is derived in [16,15,1]. The stability for nonlinear-switched systems is proposed in [4,2,3] and the linearization method is developed in [2]. Multiple Lyapunov function techniques are used in [10–20] to investigate the stability of the switched systems. The stability of some slow-switched control systems is studied in [12–25]. Lagrange stability for switched systems is considered in [4,3,24]. A review of stability results covering the majority of research works can be found in [6]. Stability analysis of switched systems with stable and unstable subsystems is driven in [26,27].

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However, the studies for the stabilities of switched systems have mainly focused on the systems without time delay due to its difficulty. In this paper, we study stability properties of linear and nonlinear switched delay systems with stable and unstable subsystems. Firstly, we prove some useful inequalities in Section 2. Then switching laws are proposed in Section 3, which ensures that the switched delay system is stable or exponentially stable for any switching signals under the laws. Linear switched delay systems are studied in Section 4 and some switching laws are given, which guarantee the switched system to be stable or globally exponentially stable. In Section 5, using delay inequalities instead of linearization, which may lose some nonlinear properties, we study a nonlinear switched system and propose a switching law that guarantees the stability or local exponential stability of the systems. Some examples are given to illustrate the main results. Section 6 concludes the paper.

## 2. Preliminary

In this section, some useful notations are given, and necessary inequalities are derived.

Let  $R$  be the set of real numbers,  $R_+$  the set of nonnegative numbers,  $R^n$  the  $n$ -dimensional space,  $R^{n \times m}$  the  $n \times m$  matrix space, respectively. If  $A = (a_{ij}) \in R^{n \times m}$ , denote  $\|A\| = \max_{1 \leq j \leq m} \sum_{i=1}^n |a_{ij}|$  the norm of  $A$ . If  $A = (a_{ij}) \in R^{n \times n}$ , denote  $\lambda(A)$  the eigenvalue of  $A$ ,  $\text{Re}\lambda(A)$  the real part of  $\lambda(A)$ .

Let  $C_\tau = \{\phi : \phi \in C[-\tau, 0], R^n\}$ , where  $\tau > 0$  is a constant. If  $x \in C_\tau$ , define  $x_t = x(t + \theta)$ ,  $\theta \in [-\tau, 0]$  and  $\|x_t\| = \sup_{-\tau \leq s \leq t} \|x(s)\|$ .

Denote

$$\Gamma(\lambda, t - s) = \begin{cases} 1, & t \leq s, \\ e^{\lambda(t-s)}, & t \geq s; \end{cases}$$

$$K_1 = \{\phi | \phi \in C[R_+, R_+], \phi(0) = 0, \phi \text{ is nondecreasing}\}.$$

$$\aleph = \{1, 2, \dots\},$$

$$\aleph_i = \{1, 2, \dots, N_i\}.$$

In this paper, we study the stability of the following switched delay system

$$x'(t) = f_{\sigma_i}(t, x(t), x_t), \quad \sigma_i \in \aleph_i, \quad (2.1)$$

where  $x \in R^n$ ,  $f_{\sigma_i} \in C[R \times R^n \times C_\tau, R^n]$ ,  $f_{\sigma_i}(t, 0, 0) = 0$  for any  $\sigma_i \in \aleph_i$ ,  $i = 1, 2$ .

In the rest of the paper, let  $i_k \in \aleph_1 \cup \aleph_2$  denote the  $i_k$ th subsystem,  $t_k$  the switching points,  $[t_k, t_{k+1})$  the time period over which the  $i_k$ th subsystem is activated. Let  $x_{\sigma_1}(t)$  and  $x_{\sigma_2}(t)$  denote the solution of systems

$$x'(t) = f_{\sigma_1}(t, x(t), x_t), \quad \sigma_1 \in \aleph_1, \quad (2.2)$$

and

$$x'(t) = f_{\sigma_2}(t, x(t), x_t), \quad \sigma_2 \in \aleph_2, \quad (2.3)$$

respectively, i.e.,

$$x'_{\sigma_1}(t) = f_{\sigma_1}(t, x_{\sigma_1}(t), x_{\sigma_1 t}), \quad \sigma_1 \in \aleph_1,$$

$$x'_{\sigma_2}(t) = f_{\sigma_2}(t, x_{\sigma_2}(t), x_{\sigma_2 t}), \quad \sigma_2 \in \aleph_2.$$

Let  $k_{\sigma_i}(t)$  be the number of times that the  $\sigma_i$ th subsystem of (2.1) is switched on during  $[t_0, t)$ ,  $\pi_{\sigma_i}(t)$  be the total activation time of  $\sigma_i$ th subsystem of (2.1) during  $[t_0, t)$ .

Assume that

(B1)  $t_{k+1} - t_k \geq \tau$ ,  $\lim_{k \rightarrow \infty} t_k = \infty$ ;

(B2) The switching sequence is minimal, i.e.,  $i_k \neq i_{k+1}$ ;

(B3) Each  $f_{\sigma_i}(t, x, y)$  is locally Lipschitzian in  $x$  and  $y$ .

Our aim is to find switching laws under which switched delay system (2.1) is stable or exponentially stable. Assumption (B1) is introduced to prevent the problem from being trivial. In fact, if after  $t \geq t_k$  the system (2.1) is not switched any more, then the switched system becomes

$$x'_{i_k}(t) = f_{i_k}(t, x_{i_k}(t), x_{i_k t}), \quad t \geq t_k$$

and the convergent property of (2.1) is trivially dependent on the  $i_k$ th subsystem. Assumption (B2) means that for any consecutive interval  $[t_{k-1}, t_k)$ ,  $[t_k, t_{k+1})$ , the active subsystems are different. The assumption (B3) guarantees that all subsystems have a unique solution for any initial condition.

**Definition 2.1.** For given switching signal, the equilibrium point of system (2.1) is said to be

- (i) stable, if for any given  $t_0$  and  $\epsilon > 0$ , there exists a  $\delta > 0$  such that  $\|x_{t_0}\| < \delta$  implies that  $\|x(t; t_0, x_{t_0})\| \leq \epsilon$  for  $t \geq t_0$ . Otherwise, the system is said to be unstable;
- (ii) locally exponentially stable, if there exist  $M > 0$ ,  $\gamma > 0$  and  $\epsilon > 0$  such that if  $\|x_{t_0}\| < \gamma$ , the solution  $x(t; t_0, x_{t_0})$  of (2.1) satisfies

$$\|x(t; t_0, x_{t_0})\| \leq M e^{-\epsilon(t-t_0)}, \quad t \geq t_0;$$

- (iii) globally exponentially stable, if there exist  $M > 0$  and  $\epsilon > 0$  such that the solution  $x(t; t_0, x_{t_0})$  of (2.1) satisfies

$$\|x(t; t_0, x_{t_0})\| \leq M \|x_{t_0}\| e^{-\epsilon(t-t_0)}, \quad t \geq t_0.$$

**Theorem 2.1.** Assume that  $\alpha > 0$ ,  $V \in C^1[R, R_+]$ ,  $g \in K_1$ ,  $h(s, \cdot) \in K_1$  for any fixed  $s \in R$ , and

$$V'(t) \leq -\alpha V(t) + g(\|V_t\|)\|V_t\| + \int_{t_0}^t h(t-s, V(s))V(s)ds.$$

If there exist constants  $r > 0$  and  $\lambda > 0$  such that

$$g(r) + \int_0^\infty h(s, r)e^{\lambda s}ds < \alpha,$$

then there exists an  $\epsilon > 0$  such that  $\|V_{t_0}\| < r$  implies that

$$V(t) \leq \|V_{t_0}\| e^{-\epsilon(t-t_0)}, \quad t \geq t_0,$$

where  $\epsilon$  satisfies

$$\epsilon + g(r)e^{\epsilon\tau} + \int_0^\infty h(s, r)e^{\epsilon s}ds < \alpha.$$

**Proof.** Since  $g(r) + \int_0^\infty h(s, r)e^{\lambda s}ds < \alpha$ , there exists an  $\epsilon$ ,  $0 < \epsilon \leq \lambda$ , such that

$$g(r)e^{\epsilon\tau} + \int_0^\infty h(s, r)e^{\epsilon s}ds < \alpha - \epsilon.$$

Let

$$P(t) = V(t)\Gamma(\epsilon, t - t_0).$$

Then  $P(t) \geq V(t)$ ,  $\|P_t\| \geq \|V_t\|$  and for  $t > t_0$ ,

$$\begin{aligned} P'(t) &= V'(t)\Gamma(\epsilon, t - t_0) + \epsilon V(t)\Gamma(\epsilon, t - t_0) \\ &\leq \Gamma(\epsilon, t - t_0) \left[ -\alpha V(t) + g(\|V_t\|)\|V_t\| + \int_{t_0}^t h(t-s, V(s))V(s)ds \right] + \epsilon V(t)\Gamma(\epsilon, t - t_0) \\ &\leq -\alpha P(t) + g(\|V_t\|)\|P_t\|e^{\epsilon\tau} + \int_{t_0}^t h(t-s, V(s))P(s)e^{\epsilon(t-s)}ds + \epsilon P(t) \\ &\leq (-\alpha + \epsilon)P(t) + g(\|V_t\|)\|P_t\|e^{\epsilon\tau} + \int_{t_0}^t h(t-s, V(s))P(s)e^{\epsilon(t-s)}ds. \end{aligned} \quad (2.4)$$

We claim that  $\|P_{t_0}\| < r$  implies  $P(t) \leq \|P_{t_0}\|$ , which can be justified as follows.

For any  $d \in (1, r/\|P_{t_0}\|)$ ,  $P(t) \leq d\|P_{t_0}\| = d\|V_{t_0}\| =: M$ .

If it is not true, then there must exist a  $t^* > t_0$ , such that

$$P(t^*) = M, \quad P(t) < M, \quad t < t^*.$$

Thus  $P'(t^*) \geq 0$ . On the other hand, it follows from (2.4) that

$$\begin{aligned} P'(t^*) &\leq (-\alpha + \epsilon)P(t^*) + g(\|V_{t^*}\|)\|P_{t^*}\|e^{\epsilon\tau} + \int_{t_0}^{t^*} h(t^* - s, V(s))P(s)e^{\epsilon(t^*-s)}ds \\ &\leq (-\alpha + \epsilon)M + g(\|P_{t^*}\|)e^{\epsilon\tau}M + \int_{t_0}^{t^*} h(t^* - s, P(s))Me^{\epsilon(t^*-s)}ds \\ &\leq M \left[ (-\alpha + \epsilon) + g(r)e^{\epsilon\tau} + \int_0^\infty h(s, r)e^{\epsilon s}ds \right] < 0, \end{aligned}$$

this contradiction implies  $P(t) \leq d\|P_{t_0}\|$ . Let  $d \rightarrow 1$ , we obtain  $P(t) \leq \|P_{t_0}\|$  and thus

$$V(t) \leq \|V_{t_0}\|e^{-\epsilon(t-t_0)}.$$

The proof is complete.  $\square$

### 3. Switching laws

In this section, we first prove an inequality for switched delay system (2.1). Based on the inequality, switching laws are derived which guarantee stability of switched delay system (2.1).

**Theorem 3.1.** Assume that there exist  $M_{\sigma_i}, \epsilon_{\sigma_i} > 0$  such that

$$\|x_{\sigma_1}(t)\| \leq M_{\sigma_1}\|x_{\sigma_1 t_0}\|e^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t \geq t_0, \quad \sigma_1 \in \aleph_1, \quad (3.1)$$

and

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2}\|x_{\sigma_2 t_0}\|e^{\epsilon_{\sigma_2}(t-t_0)}, \quad t \geq t_0, \quad \sigma_2 \in \aleph_2. \quad (3.2)$$

Then for any function  $x_{t_0} \in C_\tau$ , the solution  $x(t)$  of switched delay system (2.1) satisfies

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \aleph_i} M_{\sigma_i}^{k_{\sigma_i}} \exp \left\{ \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} [k_{\sigma_1}(t)\tau - \pi_{\sigma_1}(t)] + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t) \right\}. \quad (3.3)$$

**Proof.** We replace  $M_{\sigma_1}, M_{\sigma_2}, \epsilon_{\sigma_1}, \epsilon_{\sigma_2}$  with  $M_{\sigma_1,1}, M_{\sigma_2,2}, \epsilon_{\sigma_1,1}, \epsilon_{\sigma_2,2}$ . Let  $t_{j-1}, t_j$  and  $t_{j+1}$  be consecutive switching points.

- (1) Suppose that the  $l$ th subsystem of (2.2) is active on the interval  $[t_j, t_{j+1})$ ,  
 – if the  $i$ th subsystem of (2.2) is active on the interval  $[t_{j-1}, t_j)$ , then

$$\begin{aligned} \|x(t)\| &\leq M_{l,1}\|x_{t_j}\|e^{-\epsilon_{l,1}(t-t_j)} \\ &\leq M_{i,1}M_{l,1}\|x_{t_{j-1}}\|e^{-\epsilon_{i,1}(t_j-t_{j-1}-\tau)}e^{-\epsilon_{l,1}(t-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

- if the  $i$ th subsystem of (2.3) is active on the interval  $[t_{j-1}, t_j)$ , then

$$\begin{aligned} \|x(t)\| &\leq M_{l,1}\|x_{t_j}\|e^{-\epsilon_{l,1}(t-t_j)} \\ &\leq M_{l,1}M_{i,2}\|x_{t_{j-1}}\|e^{-\epsilon_{l,1}(t-t_j)}e^{\epsilon_{i,2}(t_{j-1}-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

- (2) Suppose that the  $l$ th subsystem of (2.3) is active on the interval  $[t_j, t_{j+1})$ ,  
 – if the  $i$ th subsystem of (2.2) is active on the interval  $[t_{j-1}, t_j)$ , then

$$\begin{aligned} \|x(t)\| &\leq M_{l,2}\|x_{t_j}\|e^{\epsilon_{l,2}(t-t_j)} \\ &\leq M_{i,1}M_{l,2}\|x_{t_{j-1}}\|e^{-\epsilon_{i,1}(t_{j-1}-t_j-\tau)}e^{\epsilon_{l,2}(t-t_j)}, \quad t \in [t_j, t_{j+1}); \end{aligned}$$

- if the  $i$ th subsystem of (2.3) is active on the interval  $[t_{j-1}, t_j)$ , then

$$\begin{aligned} \|x(t)\| &\leq M_{l,2}\|x_{t_j}\|e^{\epsilon_{l,2}(t-t_j)} \\ &\leq M_{i,2}M_{l,2}\|x_{t_{j-1}}\|e^{\epsilon_{i,2}(t_{j-1}-t_j)}e^{\epsilon_{l,2}(t-t_j)}, \quad t \in [t_j, t_{j+1}). \end{aligned}$$

Using iterative method, we can obtain

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \aleph_i} M_{\sigma_i}^{k_{\sigma_i}} \exp \left\{ \sum_{\sigma_1 \in \aleph_{\sigma_1}} \epsilon_{\sigma_1} [k_{\sigma_1}(t)\tau - \pi_{\sigma_1}(t)] + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t) \right\}.$$

The proof is complete.  $\square$

Specially, if  $\aleph_1 = \aleph_2 = \{1\}$ , then (2.1) becomes

$$x'(t) = f_{\sigma}(t, x(t), x_t) \quad (3.4)$$

where  $\sigma = 1, 2$ . Let  $x_1(t)$  and  $x_2(t)$  be the solution of system, then

$$x'_1(t) = f_1(t, x_1(t), x_{1t}) \quad (3.5)$$

$$x'_2(t) = f_2(t, x_2(t), x_{2t}) \quad (3.6)$$

respectively,  $k(t)$  be the times of subsystem (3.5) being switched on during the interval  $[t_0, t)$ ,  $\pi_1(t)$  and  $\pi_2(t)$  are the total time of subsystems (3.5), (3.6) used respectively on  $[t_0, t)$ , then we have

**Corollary 3.1.** Assume that there exist  $M_1, M_2 > 0$  and  $\epsilon_1, \epsilon_2 > 0$ , such that

$$\|x_1(t)\| \leq M_1 \|x_{1t_0}\| e^{-\epsilon_1(t-t_0)}, \quad t \geq t_0,$$

$$\|x_2(t)\| \leq M_2 \|x_{2t_0}\| e^{\epsilon_2(t-t_0)}, \quad t \geq t_0.$$

Then the solution of switched delay system (3.4) satisfies

$$\|x(t)\| \leq \|x_{t_0}\| (M_1 M_2)^{k(t)} \exp\{\epsilon_1 k(t)\tau - \epsilon_1 \pi_1(t) + \epsilon_2 \pi_2(t)\}. \quad (3.7)$$

Define switching laws as follows:

$$(S_1) \quad \frac{\pi_2(t)}{\pi_1(t)} + \frac{k(t) \ln M_1 M_2}{\epsilon_2 \pi_1(t)} + \frac{k(t) \epsilon_1 \tau}{\epsilon_2 \pi_1(t)} \leq \frac{\epsilon_1}{\epsilon_2}, \quad t \geq T \text{ for some } T;$$

$$(S_2) \quad \limsup_{t \rightarrow \infty} \left[ \frac{\pi_2(t)}{\pi_1(t)} + \frac{k(t) \ln M_1 M_2}{\epsilon_2 \pi_1(t)} + \frac{k(t) \epsilon_1 \tau}{\epsilon_2 \pi_1(t)} \right] = \frac{\epsilon_1}{\epsilon_2};$$

$$(S_3) \quad \limsup_{t \rightarrow \infty} \left[ \frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t) \ln M_1 M_2}{\epsilon_2 \pi_1(t)} + \frac{k(t) \epsilon_1 \tau}{\epsilon_2 \pi_1(t)} \right] < 0.$$

We can prove the following theorems.

**Theorem 3.2.** Assume that the conditions of Corollary 3.1 hold. Then

- (1) switching laws (S<sub>1</sub>) and (S<sub>2</sub>) imply that switched delay system (3.4) is stable;
- (2) switching laws (S<sub>3</sub>) implies that switched delay system (3.4) is exponentially stable.

**Proof.** Without loss of generality, suppose that switched delay system (3.4) is

$$\begin{cases} x'(t) = f_1(t, x, x_t), & t \in [t_{2k}, t_{2k+1}), \\ x'(t) = f_2(t, x, x_t), & t \in [t_{2k-1}, t_{2k}), \end{cases}$$

where  $t_j$  are switching points,  $j \in \aleph$ . Then, when  $t \in [t_{2k}, t_{2k+1})$

$$\begin{aligned} \|x(t)\| &\leq M_1 \|x_{t_{2k}}\| e^{-\epsilon_1(t-t_{2k})} \\ &\leq M_1 M_2 \|x_{t_{2k-1}}\| e^{\epsilon_2(t_{2k}-t_{2k-1})} e^{-\epsilon_1(t-t_{2k})} \\ &\leq M_1 M_2 M_1 \|x_{t_{2k-2}}\| e^{\epsilon_2(t_{2k}-t_{2k-1})} e^{-\epsilon_1[(t-t_{2k})+(t_{2k-1}-t_{2k-2}-\tau)]} \\ &\leq (M_1 M_2)^2 \|x_{t_{2k-3}}\| e^{\epsilon_2[(t_{2k}-t_{2k-1})+(t_{2k-2}-t_{2k-3})]} e^{-\epsilon_1[(t-t_{2k})+(t_{2k-1}-t_{2k-2}-\tau)]} \\ &\leq \dots \dots \dots \\ &\leq (M_1 M_2)^k(t) \|x_{t_0}\| e^{\epsilon_2 \pi_2(t)} e^{-\epsilon_1 \pi_1(t) + k(t) \epsilon_1 \tau} \\ &= \|x_{t_0}\| e^{\epsilon_2 \pi_2(t) - \epsilon_1 \pi_1(t) + k(t) \epsilon_1 \tau + k(t) \ln M_1 M_2}. \end{aligned}$$

So

$$\|x(t)\| \leq \|x_{t_0}\| e^{\epsilon_2 \pi_1(t) \left[ \frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)} + \frac{k(t)\ln M_1 M_2}{\epsilon_2\pi_1(t)} \right]}.$$

It is easy to see that the result (1) of [Theorem 3.2](#) is true from about inequality. Next, we prove the result (2).

Let

$$\limsup_{t \rightarrow \infty} \left[ \frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\ln M_1 M_2}{\epsilon_2\pi_1(t)} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)} \right] = -2\epsilon^*.$$

Without loss of generality, suppose that

$$\frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\ln M_1 M_2}{\epsilon_2\pi_1(t)} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)} < -\epsilon^*, \quad t \geq t_0.$$

Then  $\pi_1(t) \geq \frac{\epsilon_2}{\epsilon_1} \pi_2(t)$ , and

$$\begin{aligned} \epsilon_2 \pi_1(t) \left[ \frac{\pi_2(t)}{\pi_1(t)} - \frac{\epsilon_1}{\epsilon_2} + \frac{k(t)\epsilon_1\tau}{\epsilon_2\pi_1(t)} + \frac{k(t)\ln M_1 M_2}{\epsilon_2\pi_1(t)} \right] &\leq -\epsilon^* \epsilon_2 \pi_1(t) \\ &\leq -\epsilon^* \epsilon_2 \pi_1(t)/2 - \epsilon^* \epsilon_2 \pi_1(t)/2 \\ &\leq -\frac{\epsilon^* \epsilon_2}{2} \pi_1(t) - \frac{\epsilon^* \epsilon_2^2}{2\epsilon_1} \pi_2(t) \\ &\leq -\epsilon [\pi_1(t) + \pi_2(t)] = -\epsilon(t - t_0), \end{aligned}$$

where  $\epsilon = \min\{\frac{\epsilon^* \epsilon_2}{2}, \frac{\epsilon^* \epsilon_2^2}{2\epsilon_1}\}$ . Thus

$$\|x(t)\| \leq \|x_{t_0}\| e^{-\epsilon(t-t_0)}.$$

The proof is complete.  $\square$

For switched delay system (2.1), define switching law as following:

(S<sub>4</sub>)  $\sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t) \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] \leq \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)$ ,  $t \geq T$  for some  $T$ ;

(S<sub>5</sub>)  $\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t) \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] = 1$ ;

(S<sub>6</sub>)  $\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \aleph_i} [k_{\sigma_i}(t) \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] < 1$ .

**Theorem 3.3.** Assume that the conditions of [Theorem 3.1](#) hold. Then the switched systems (2.1) is

(1) switching laws (S<sub>4</sub>) and (S<sub>5</sub>) imply that switched delay system (2.1) is stable;

(2) switching laws (S<sub>6</sub>) implies that switched delay system (2.1) is exponentially stable.

The proof is similar to that of [Theorem 3.2](#).

#### 4. Linear switched delay systems

In this section, we study the linear switched delay systems

$$x'(t) = A_{\sigma_i} x(t) + B_{\sigma_i} x(t - \tau(t)) + \int_{t_0}^t h_{\sigma_i}(t-s)x(s)ds, \quad (4.1)$$

where  $x \in R^n$ ,  $A_{\sigma_i}, B_{\sigma_i} \in R^{n \times n}$ ,  $h_{\sigma_i} \in C[R_+, R^n]$ ,  $\sigma_i \in \aleph_{\sigma_i}$ ,  $i = 1, 2$ ,  $0 \leq \tau(t) \leq \tau$ .

Without loss of generality, we make the following assumptions.

(A1)  $\Phi_{\sigma_i} \in R^{n \times n}$ , satisfy

$$\begin{cases} \frac{\partial \Phi_{\sigma_i}(t, s)}{\partial t} = A_{\sigma_i} \Phi_{\sigma_i}(t, s), \\ \Phi_{\sigma_i}(s, s) = I, \end{cases}$$

where  $I$  is an identity matrix, and  $\sigma_i \in \aleph_i$ ,  $i = 1, 2$ .

(A2) there exist  $\lambda_{\sigma_i} > 0$  and  $M_{\sigma_i} \geq 1$ , such that

$$\|\Phi_{\sigma_1}(t, t_0)\| \leq M_{\sigma_1} e^{-\lambda_{\sigma_1}(t-t_0)},$$

and

$$\|\Phi_{\sigma_2}(t, t_0)\| \leq M_{\sigma_2} e^{\lambda_{\sigma_2}(t-t_0)}.$$

Let  $x_{\sigma_1}(t)$  and  $x_{\sigma_2}(t)$  be the solutions of the subsystems

$$x'_{\sigma_1}(t) = A_{\sigma_1}x_{\sigma_1} + B_{\sigma_1}x_{\sigma_1}(t - \tau(t)) + \int_{t_0}^t h_{\sigma_1}(t-s)x_{\sigma_1}(s)ds, \quad (4.2)$$

$$x'_{\sigma_2}(t) = A_{\sigma_2}x_{\sigma_2} + B_{\sigma_2}x_{\sigma_2}(t - \tau(t)) + \int_{t_0}^t h_{\sigma_2}(t-s)x_{\sigma_2}(s)ds, \quad (4.3)$$

then we have the following theorems.

**Theorem 4.1.** Assume that (A1), (A2) hold and  $\exists \epsilon_{\sigma_i} > 0$ ,  $i = 1, 2$ , such that

$$-\lambda_{\sigma_1} + M_{\sigma_1} \|B_{\sigma_1}\| e^{\epsilon_{\sigma_1} \tau} + \int_0^\infty M_{\sigma_1} \|h_{\sigma_1}(s)\| e^{\epsilon_{\sigma_1} s} ds < -\epsilon_{\sigma_1},$$

$$\lambda_{\sigma_2} + M_{\sigma_2} \|B_{\sigma_2}\| + \int_0^\infty M_{\sigma_2} \|h_{\sigma_2}(s)\| e^{-\epsilon_{\sigma_2} s} ds < \epsilon_{\sigma_2}.$$

Then switched delay systems (4.1) is

(1) stable, if  $\exists T > 0$ , such that

$$\frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] \leq 1, \quad t \geq T \quad (4.4)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] = 1, \quad (4.5)$$

(2) globally exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [k_{\sigma_i} \ln M_{\sigma_i} + \epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] < 1. \quad (4.6)$$

**Proof.** By the variation of parameters, the solution of subsystem (4.2) is

$$x_{\sigma_1}(t) = \Phi_{\sigma_1}(t, t_0)x_{\sigma_1}(t_0) + \int_{t_0}^t \Phi_{\sigma_1}(t, s)B_{\sigma_1}x_{\sigma_1}(s - \tau(s))ds + \int_{t_0}^t \Phi_{\sigma_1}(t, s) \int_{t_0}^s h_{\sigma_1}(s - \theta)x_{\sigma_1}(\theta)d\theta ds.$$

Taking the norm on both sides, we have

$$\begin{aligned} \|x_{\sigma_1}(t)\| &\leq \|\Phi_{\sigma_1}(t, t_0)\| \|x_{\sigma_1}(t_0)\| + \int_{t_0}^t \|\Phi_{\sigma_1}(t, s)\| \|B_{\sigma_1}\| \|x_{\sigma_1}(s - \tau(s))\| ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \|\Phi_{\sigma_1}(t, s)\| \|h_{\sigma_1}(s - \theta)\| \|x_{\sigma_1}(\theta)\| d\theta ds \\ &\leq M_{\sigma_1} \|x_{\sigma_1 t_0}\| e^{-\lambda_{\sigma_1}(t-t_0)} + \int_{t_0}^t M_{\sigma_1} e^{-\lambda_{\sigma_1}(t-s)} \|B_{\sigma_1}\| \|x_{\sigma_1 s}\| ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s M_{\sigma_1} e^{-\lambda_{\sigma_1}(t-s)} \|h_{\sigma_1}(s - \theta)\| \|x_{\sigma_1}(\theta)\| d\theta ds \\ &=: P_{\sigma_1}^*(t), \quad t \geq t_0. \end{aligned}$$

Let

$$P_{\sigma_1}(t) = \begin{cases} P_{\sigma_1}^*(t), & t \geq t_0, \\ M_{\sigma_1} \|x_{\sigma_1 t_0}\|, & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then  $P_{\sigma_1}(t) \geq \|x_{\sigma_1}(t)\|$ ,  $t \geq t_0 - \tau$  and

$$\begin{aligned} P'_{\sigma_1}(t) &= -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1} \|B_{\sigma_1}\| \|x_{\sigma_1 t}\| + \int_{t_0}^t M_{\sigma_1} \|h_{\sigma_1}(t-s)\| \|x_{\sigma_1}(s)\| ds \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1} \|B_{\sigma_1}\| \|P_{\sigma_1 t}\| + \int_{t_0}^t M_{\sigma_1} \|h_{\sigma_1}(t-s)\| P_{\sigma_1}(s) ds. \end{aligned}$$

For any  $K > 1$ , we claim that

$$P_{\sigma_1}(t) < K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}. \quad (4.7)$$

In fact, if (4.7) is not true, then  $\exists t^* > t_0$ , such that

$$\begin{aligned} P_{\sigma_1}(t^*) &= K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t^*-t_0)}, \quad \text{and} \\ P_{\sigma_1}(t) &\leq K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t < t^*. \end{aligned}$$

Thus  $P'_{\sigma_1}(t^*) \geq -\epsilon_{\sigma_1} K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t^*-t_0)} = -\epsilon_{\sigma_1} P_{\sigma_1}(t^*)$ . On the other hand

$$\begin{aligned} P'_{\sigma_1}(t^*) &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1} \|B_{\sigma_1}\| \|P_{\sigma_1 t^*}\| + \int_{t_0}^{t^*} M_{\sigma_1} \|h_{\sigma_1}(t^*-s)\| P_{\sigma_1}(s) ds \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1} \|B_{\sigma_1}\| P_{\sigma_1}(t^*) e^{\epsilon_{\sigma_1} \tau} + \int_{t_0}^{t^*} K M_{\sigma_1} \|h_{\sigma_1}(t^*-s)\| \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(s-t_0)} ds \\ &= P_{\sigma_1}(t^*) \left[ -\lambda_{\sigma_1} + M_{\sigma_1} \|B_{\sigma_1}\| e^{\epsilon_{\sigma_1} \tau} + \int_{t_0}^{t^*} M_{\sigma_1} \|h_{\sigma_1}(t^*-s)\| e^{\epsilon_{\sigma_1}(t^*-s)} ds \right] < -\epsilon_{\sigma_1} P_{\sigma_1}(t^*). \end{aligned}$$

This contradiction implies that  $P_{\sigma_1}(t) < K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}$ . Let  $K \rightarrow 1$ , we obtain  $P_{\sigma_1}(t) \leq \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}$  and hence

$$\|x_{\sigma_1}(t)\| \leq M_{\sigma_1} \|x_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}. \quad (4.8)$$

Similarly, we can obtain

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2} \|x_{\sigma_2 t_0}\| e^{\epsilon_{\sigma_2}(t-t_0)}. \quad (4.9)$$

Combining (4.8) and (4.9) and Theorem 3.1, we have

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \mathbb{N}_i} M_{\sigma_i}^{k_{\sigma_i}(t)} \exp \left\{ \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau - \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t) + \sum_{\sigma_2 \in \mathbb{N}_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t) \right\}.$$

The results of (1) and (2) of Theorem 4.1 can be obtained by Theorem 3.2. The proof is complete.  $\square$

**Remark 1.** It should be pointed out that, in Theorem 4.1, (4.4)–(4.6) are actually switching laws which guarantee the switched delay system (4.1) is stable or locally exponentially stable.

**Corollary 4.1.** Assume that  $\exists \lambda_{\sigma_i} > 0$ ,  $\epsilon_{\sigma_2} > 0$  such that

- (i)  $\lambda(A_{\sigma_1}^T + A_{\sigma_1}) \leq -2\lambda_{\sigma_1}$ ,  $\lambda(A_{\sigma_2}^T + A_{\sigma_2}) \leq 2\lambda_{\sigma_2}$ ;
- (ii)  $-\lambda_{\sigma_1} + \|B_{\sigma_1}\| e^{\epsilon_{\sigma_1} \tau} + \int_0^\infty \|h_{\sigma_1}(s)\| e^{\epsilon_{\sigma_1} s} ds < -\epsilon_{\sigma_1}$ ;
- (iii)  $\lambda_{\sigma_2} + \|B_{\sigma_2}\| + \int_0^\infty \|h_{\sigma_2}(s)\| e^{-\epsilon_{\sigma_2} s} ds < \epsilon_{\sigma_2}$ .

Then the switched delay systems (4.1) is

(1) stable, if  $\exists T > 0$ , such that

$$\frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [\epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] \leq 1, \quad t \geq T$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [\epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] = 1,$$

(2) globally exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{1}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \sum_{\sigma_i \in \mathbb{N}_i} [\epsilon_{\sigma_1} k_{\sigma_1} \tau + \epsilon_{\sigma_2} \pi_{\sigma_2}(t)] < 1.$$

**Proof.** Let  $V_{\sigma_i} = x_{\sigma_i}^T x_{\sigma_i}$ , then the derivative of  $V_{\sigma_i}$  along the system  $x'_{\sigma_i} = A_{\sigma_i} x_{\sigma_i}$  is

$$V'_{\sigma_i} = x_{\sigma_i}^T (A_{\sigma_i}^T + A_{\sigma_i}) x_{\sigma_i}.$$

Thus

•

$$V'_{\sigma_1}(t) \leq -2\lambda_{\sigma_1} V_{\sigma_1}(t) \Rightarrow V_{\sigma_1}(t) \leq V_{\sigma_1}(t_0) e^{-2\lambda_{\sigma_1}(t-t_0)}, \quad \text{and} \quad \|\Phi_{\sigma_1}(t, t_0)\| \leq e^{-\lambda_{\sigma_1}(t-t_0)}; \quad (4.10)$$

•

$$V'_{\sigma_2}(t) \leq 2\lambda_{\sigma_2} V_{\sigma_2}(t) \Rightarrow V_{\sigma_2}(t) \leq V_{\sigma_2}(t_0) e^{2\lambda_{\sigma_2}(t-t_0)}, \quad \text{and} \quad \|\Phi_{\sigma_2}(t, t_0)\| \leq e^{\lambda_{\sigma_2}(t-t_0)}. \quad (4.11)$$

All the results of this theorem can be obtained by (4.10) and (4.11) and Theorem 4.1. The proof is complete.  $\square$

We illustrate the results of Corollary 4.1 with a simple linear switched delay system.

**Example 4.1.**

$$A_1 = \begin{bmatrix} -3 & 10 \\ -10 & -3.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} \frac{1}{2e} & 0 \\ \frac{1}{e} & 0 \end{bmatrix}, \quad h_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -1 & -3 \\ 3 + \sqrt{12} & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \frac{1}{3e} & \frac{1}{4e} \\ 0 & 0 \end{bmatrix}, \quad h_2 = \begin{bmatrix} e^t \cos^2 t & 0 \\ e^t \sin^2 t & 0 \end{bmatrix}.$$

Hence  $\lambda(A_1^T + A_1) \leq -6$ ,  $\lambda(A_2^T + A_2) \leq 4$ ;  $\|B_1\| = \frac{3}{2e}$ ,  $\|B_2\| = \frac{1}{3e}$ ;  $\|h_1\| = 0$ ,  $\|h_2\| = e^t$ . Take  $\tau = 1$ ,  $\lambda_1 = 3$ ,  $\lambda_2 = 2$ ,  $\epsilon_1 = 1$ ,  $\epsilon_2 = 4$ .

Then,

$$-\lambda_1 + \|B_1\| e^{\epsilon_1 \tau} + \int_0^\infty \|h_1(s)\| e^{\epsilon_1 s} ds = -3 + 3/2 < -1 = -\epsilon_1;$$

$$\lambda_2 + 1 + \|B_2\| e^\tau + \int_0^\infty \|h_2(s)\| e^{(-\epsilon_2 + 1)s} ds < 2 + 1 + 1/3 + 1/2 \leq 4 = \epsilon_2.$$

With the switching laws from Corollary 4.1, we have

$$(S_1) \quad \pi_1(t) > k_1(t) + 4\pi_{\sigma_2}(t) \text{ and } \limsup_{t \rightarrow \infty} \frac{k_1(t) + 4\pi_{\sigma_2}(t)}{\pi_1(t)} = 1$$

$$(S_2) \quad \limsup_{t \rightarrow \infty} \frac{k_1(t) + 4\pi_{\sigma_2}(t)}{\pi_1(t)} < 1.$$

By Corollary 4.1, we know that

- if we choose the switching law  $(S_1)$ , then the switched delay system

$$x'(t) = A_i x(t) + B_i x(t - \tau(t)) + \int_{t_0}^t h_i(t - s)x(s)ds, \quad (4.12)$$

where  $i = 1, 2$ ,  $0 \leq \tau(t) \leq 1$ , is stable;

- if we choose the switching law  $(S_2)$ , then switched delay system (4.12) is globally exponentially stable.

## 5. Nonlinear switched delay systems

In this section, we study the nonlinear switched delay systems

$$x'(t) = A_{\sigma_i} x + f_{\sigma_i}(t, x_t) + \int_{t_0}^t h_{\sigma_i}(t - s, x(s))ds, \quad (5.1)$$

where  $x \in R^n$ ,  $A_{\sigma_i} \in R^{n \times n}$ ,  $f_{\sigma_i} \in C[R \times C_\tau, R^n]$ ,  $h_{\sigma_i} \in C[R \times R^n, R^n]$ ,  $\sigma_i \in \aleph_i$ ,  $i = 1, 2$ .

Denote  $\hat{f}_{\sigma_2}(t, y_t) = e^{-\epsilon_{\sigma_2}(t-t_0)} f_{\sigma_2}(t, x_t)$ ,  $\hat{h}_{\sigma_2}(t - s, y(s)) = e^{-\epsilon_{\sigma_2}(t-t_0)} h_{\sigma_2}(t - s, x(s))$ , where  $y(s) = x(s)e^{-\epsilon_{\sigma_2}(s-t_0)}$ , then we can obtain the following results about local exponential stability.

**Theorem 5.1.** Assume that (A1), (A2) hold and

- (i) there exist  $\tilde{f}_{\sigma_i} \in K_1$  and  $\tilde{h}_{\sigma_i} \in C[R^2, R_+]$  such that  $\tilde{h}_{\sigma_i}(s, \cdot) \in K_1$  for any  $s \in R_+$  and
- $$\|f_{\sigma_1}(t, x_t)\| \leq \tilde{f}_{\sigma_1}(\|x_t\|)\|x_t\|, \quad \|\hat{f}_{\sigma_2}(t, x_t)\| \leq \tilde{f}_{\sigma_2}(\|x_t\|)\|x_t\|,$$
- $$\|h_{\sigma_1}(t - s, x(s))\| \leq \tilde{h}_{\sigma_1}(t - s, \|x(s)\|)\|x(s)\|, \quad \|\hat{h}_{\sigma_2}(t - s, x(s))\| \leq \tilde{h}_{\sigma_2}(t - s, \|x(s)\|)\|x(s)\|;$$
- (ii)  $\exists \epsilon_{\sigma_1}, \epsilon_{\sigma_2}, r > 0$  such that

$$\epsilon_{\sigma_1} + M_{\sigma_1} \tilde{f}_{\sigma_1}(r)e^{\epsilon_{\sigma_1}\tau} + \int_0^\infty M_{\sigma_1} \tilde{h}_{\sigma_1}(s, r)e^{\epsilon_{\sigma_1}s}ds < \lambda_{\sigma_1},$$

$$\lambda_{\sigma_2} + M_{\sigma_2} \tilde{f}_{\sigma_2}(r)e^{\epsilon_{\sigma_2}\tau} + M_{\sigma_2} \int_0^\infty \tilde{h}_{\sigma_2}(s, r)e^{\epsilon_{\sigma_2}s}ds < \epsilon_{\sigma_2}.$$

Then the switched delay systems (5.1) is

- (1) stable, if  $\exists T > 0$ , such that

$$\frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \leq 1, \quad t \geq T \quad (5.2)$$

and

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} = 1; \quad (5.3)$$

- (2) locally exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} k_{\sigma_1}(t)\tau + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} < 1. \quad (5.4)$$

**Proof.** Let  $x_{\sigma_1}(t)$  be the solution of the  $\sigma_1$ th subsystem,  $\sigma_1 \in \aleph_1$ , i.e.

$$x'_{\sigma_1}(t) = A_{\sigma_1} x_{\sigma_1} + f_{\sigma_1}(t, x_{\sigma_1 t}) + \int_{t_0}^t h_{\sigma_1}(t - s, x_{\sigma_1}(s))ds, \quad (5.5)$$

By the variation of parameters, we have

$$x_{\sigma_1}(t) = \Phi_{\sigma_1}(t, t_0)x_{\sigma_1}(t_0) + \int_{t_0}^t \Phi_{\sigma_1}(t, s)f_{\sigma_1}(t, x_{\sigma_1 s})ds + \int_{t_0}^t \Phi_{\sigma_1}(t, s) \int_{t_0}^s h_{\sigma_1}(s - \theta, x_{\sigma_1}(\theta))d\theta ds.$$

Take norm on both sides, we have

$$\begin{aligned} \|x_{\sigma_1}(t)\| &\leq \|\Phi_{\sigma_1}(t, t_0)\| \|x_{\sigma_1}(t_0)\| + \int_{t_0}^t \|\Phi_{\sigma_1}(t, s)\| \|f_{\sigma_1}(s, x_{\sigma_1 s})\| ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s \|\Phi_{\sigma_1}(t, s)\| \|h_{\sigma_1}(s - \theta, x_{\sigma_1}(\theta))\| d\theta ds \\ &\leq M_{\sigma_1} \|x_{\sigma_1 t_0}\| e^{-\lambda_{\sigma_1}(t-t_0)} + \int_{t_0}^t M_{\sigma_1} e^{-\lambda_{\sigma_1}(t-s)} \tilde{f}_{\sigma_1}(\|x_{\sigma_1 s}\|) \|x_{\sigma_1 s}\| ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s M_{\sigma_1} e^{-\lambda_{\sigma_1}(t-s)} \tilde{h}_{\sigma_1}(s - \theta, \|x_{\sigma_1}(\theta)\|) \|x_{\sigma_1}(\theta)\| d\theta ds \\ &=: P_{\sigma_1}^*(t), \quad t \geq t_0. \end{aligned}$$

Let

$$P_{\sigma_1}(t) = \begin{cases} P_{\sigma_1}^*(t), & t \geq t_0, \\ M_{\sigma_1} \|x_{\sigma_1 t_0}\|, & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then  $P_{\sigma_1}(t) \geq \|x_{\sigma_1}(t)\|$ ,  $t \geq t_0 - \tau$  and for  $t \geq t_0$

$$\begin{aligned} P'_{\sigma_1}(t) &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1} \tilde{f}_{\sigma_1}(\|x_{\sigma_1 t}\|) \|x_{\sigma_1 t}\| + \int_{t_0}^t M_{\sigma_1} \tilde{h}_{\sigma_1}(t - s, \|x_{\sigma_1}(s)\|) \|x_{\sigma_1}(s)\| ds \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t) + M_{\sigma_1} \tilde{f}_{\sigma_1}(\|P_{\sigma_1 t}\|) \|P_{\sigma_1 t}\| + \int_{t_0}^t M_{\sigma_1} \tilde{h}_{\sigma_1}(t - s, P_{\sigma_1}(s)) P_{\sigma_1}(s) ds. \end{aligned}$$

We claim that for any  $0 < \|P_{\sigma_1 t_0}\| < r$ ,  $K \in (1, \frac{r}{\|P_{\sigma_1 t_0}\|})$ ,

$$P_{\sigma_1}(t) < K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}. \quad (5.6)$$

For contradiction, suppose that  $\exists t^* > t_0$ , such that

$$\begin{aligned} P_{\sigma_1}(t^*) &= K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t^*-t_0)}, \quad \text{and} \\ P_{\sigma_1}(t) &\leq K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}, \quad t < t^*. \end{aligned}$$

Thus  $P'_{\sigma_1}(t^*) \geq -\epsilon_{\sigma_1} K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t^*-t_0)} = -\epsilon_{\sigma_1} P_{\sigma_1}(t^*)$ . On the other hand

$$\begin{aligned} P'_{\sigma_1}(t^*) &= -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1} \tilde{f}_{\sigma_1}(\|P_{\sigma_1 t^*}\|) \|P_{\sigma_1 t^*}\| + \int_{t_0}^{t^*} M_{\sigma_1} \tilde{h}_{\sigma_1}(t^* - s, P_{\sigma_1}(s)) P_{\sigma_1}(s) ds \\ &\leq -\lambda_{\sigma_1} P_{\sigma_1}(t^*) + M_{\sigma_1} \tilde{f}_{\sigma_1}(r) P_{\sigma_1}(t^*) e^{\epsilon_{\sigma_1} \tau} + \int_{t_0}^{t^*} M_{\sigma_1} \tilde{h}_{\sigma_1}(t^* - s, r) P_{\sigma_1}(t^*) e^{\epsilon_{\sigma_1}(t^*-s)} ds \\ &\leq P_{\sigma_1}(t^*) \left[ -\lambda_{\sigma_1} + M_{\sigma_1} \tilde{f}_{\sigma_1}(r) e^{\epsilon_{\sigma_1} \tau} + \int_0^\infty M_{\sigma_1} \tilde{h}_{\sigma_1}(s, r) e^{\epsilon_{\sigma_1} s} ds \right] \\ &< -\epsilon_{\sigma_1} P_{\sigma_1}(t^*). \end{aligned}$$

This contradiction implies that  $P_{\sigma_1}(t) < K \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}$ . Let  $K \rightarrow 1$ , we obtain  $P_{\sigma_1}(t) \leq \|P_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}$  and hence

$$\|x_{\sigma_1}(t)\| \leq M_{\sigma_1} \|x_{\sigma_1 t_0}\| e^{-\epsilon_{\sigma_1}(t-t_0)}. \quad (5.7)$$

For subsystems

$$x'(t) = A_{\sigma_2} x + f_{\sigma_2}(t, x_t) + \int_{t_0}^t h_{\sigma_2}(t - s, x(s)) ds, \quad (5.8)$$

let  $x_{\sigma_2}(t)$  be the solution (5.8) and let  $y_{\sigma_2}(t) = x_{\sigma_2}(t)\Gamma(-\epsilon_{\sigma_2}, t - t_0)$ , then

$$x'_{\sigma_2}(t) = A_{\sigma_2}x_{\sigma_2} + f_{\sigma_2}(t, x_{\sigma_2}(t)) + \int_{t_0}^t h_{\sigma_2}(t-s, x_{\sigma_2}(s))ds, \quad (5.9)$$

and for  $t \geq t_0$

$$\begin{aligned} y'_{\sigma_2}(t) &= (A_{\sigma_2} - \epsilon_{\sigma_2}I)y_{\sigma_2}(t) + f_{\sigma_2}(t, x_t)e^{-\epsilon_{\sigma_2}(t-t_0)} + \int_{t_0}^t h_{\sigma_2}(t-s, x(s))e^{-\epsilon_{\sigma_2}(t-t_0)}ds \\ &= (A_{\sigma_2} - \epsilon_{\sigma_2}I)y_{\sigma_2}(t) + \widehat{f}_{\sigma_2}(t, y_t) + \int_{t_0}^t \widehat{h}_{\sigma_2}(t-s, y(s))ds. \end{aligned} \quad (5.10)$$

Let  $\Phi_{\sigma_2}^*(t, t_0)$  be a matrix and satisfy

$$\begin{cases} \frac{\partial}{\partial t} \Phi_{\sigma_2}^*(t, s) = (A_{\sigma_2} - \epsilon_{\sigma_2}I) \Phi_{\sigma_2}^*(t, s) \\ \Phi_{\sigma_2}^*(s, s) = I. \end{cases}$$

Then  $\|\Phi_{\sigma_2}^*(t, s)\| \leq M_{\sigma_2}e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}$ , and the solution  $y_{\sigma_2}(t)$  of (5.10) is

$$y_{\sigma_2}(t) = \Phi_{\sigma_2}^*(t, t_0)y_{\sigma_2}(t_0) + \int_{t_0}^t \Phi_{\sigma_2}^*(t, s)\widehat{f}_{\sigma_2}(s, y_{\sigma_2}(s))ds + \int_{t_0}^t \int_{t_0}^s \Phi_{\sigma_2}^*(t, s)\widehat{h}_{\sigma_2}(s-\theta, y_{\sigma_2}(\theta))d\theta ds.$$

Thus

$$\begin{aligned} \|y_{\sigma_2}(t)\| &\leq M_{\sigma_2}\|y_{\sigma_2}(t_0)\|e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-t_0)} + \int_{t_0}^t M_{\sigma_2}e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\|\widehat{f}_{\sigma_2}(s, y_{\sigma_2}(s))\|ds \\ &\quad + \int_{t_0}^t \int_{t_0}^s M_{\sigma_2}e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\|\widehat{h}_{\sigma_2}(s-\theta, y_{\sigma_2}(\theta))\|d\theta ds \\ &\leq M_{\sigma_2}\left[\|y_{\sigma_2}(t_0)\|e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-t_0)} + \int_{t_0}^t e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\widetilde{f}_{\sigma_2}(\|y_{\sigma_2}(s)\|)\|y_{\sigma_2}(s)\|ds \right. \\ &\quad \left. + \int_{t_0}^t \int_{t_0}^s e^{(\lambda_{\sigma_2} - \epsilon_{\sigma_2})(t-s)}\widetilde{h}_{\sigma_2}(s-\theta, \|y_{\sigma_2}(\theta)\|)\|y_{\sigma_2}(\theta)\|d\theta ds\right] \\ &=: P_{\sigma_2}^*(t), \quad t \geq t_0. \end{aligned}$$

Let

$$P_{\sigma_2}(t) = \begin{cases} P_{\sigma_2}^*(t), & t \geq t_0, \\ M_{\sigma_2}\|y_{\sigma_2}(t_0)\|, & t_0 - \tau \leq t \leq t_0. \end{cases}$$

Then  $P_{\sigma_2}(t) \geq \|y_{\sigma_2}(t)\|$ ,  $t \geq t_0 - \tau$  and

$$P'_{\sigma_2}(t) \leq (\lambda_{\sigma_2} - \epsilon_{\sigma_2})P_{\sigma_2}(t) + M_{\sigma_2}\left[\widetilde{f}_{\sigma_2}(\|P_{\sigma_2}(t)\|)\|P_{\sigma_2}(t)\| + \int_{t_0}^t \widetilde{h}_{\sigma_2}(t-\theta, P_{\sigma_2}(\theta))P_{\sigma_2}(\theta)d\theta\right].$$

Using a similar approach, we can obtain  $P_{\sigma_2}(t) \leq \|P_{\sigma_2}(t)\|$  and thus

$$\|y_{\sigma_2}(t)\| \leq P_{\sigma_2}(t) \leq M_{\sigma_2}\|y_{\sigma_2}(t_0)\|.$$

Then

$$\|x_{\sigma_2}(t)\| \leq M_{\sigma_2}\|x_{\sigma_2}(t_0)\|e^{\epsilon_{\sigma_2}(t-t_0)}. \quad (5.11)$$

Combining (5.7) and (5.11) and Theorem 3.1, we have

$$\|x(t)\| \leq \|x_{t_0}\| \prod_{\sigma_i \in \mathbb{N}_i} M_{\sigma_i}^{k_{\sigma_i}(t)} \exp \left\{ \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau - \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t) + \sum_{\sigma_2 \in \mathbb{N}_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t) \right\}.$$

By Theorem 3.2, we can obtain the results of this theorem. The proof is complete.  $\square$

**Theorem 5.2.** Assume that (A1), (A2) hold and

(i) there exist  $\tilde{f}_{\sigma_1} \in K_1$  and  $\tilde{h}_{\sigma_1} \in C[R^2, R_+]$ ,  $b_{\sigma_2} \geq 0$  and  $\tilde{h}_{\sigma_2} \in C[R_+, R_+]$  such that  $\tilde{h}_{\sigma_1}(s, \cdot) \in K_1$  for any  $s \in R_+$  and

$$\begin{aligned} \|f_{\sigma_1}(t, x_t)\| &\leq \tilde{f}_{\sigma_1}(\|x_t\|)\|x_t\|, & \|f_{\sigma_2}(t, x_t)\| &\leq b_{\sigma_2}\|x_t\|, \\ \|h_{\sigma_1}(t-s, x(s))\| &\leq \tilde{h}_{\sigma_1}(t-s, \|x(s)\|)\|x(s)\|, & \|h_{\sigma_2}(t-s, x(s))\| &\leq \tilde{h}_{\sigma_2}(t-s)\|x(s)\|; \end{aligned}$$

(ii)  $\exists \epsilon_{\sigma_1}, \epsilon_{\sigma_2}, r > 0$  such that

$$\epsilon_{\sigma_1} + M_{\sigma_1} \tilde{f}_{\sigma_1}(r) e^{\epsilon_{\sigma_1} \tau} + M_{\sigma_1} \int_0^\infty \tilde{h}_{\sigma_1}(s, r) e^{\epsilon_{\sigma_1} s} ds < \lambda_{\sigma_1},$$

$$\lambda_{\sigma_2} + M_{\sigma_2} b_{\sigma_2} + M_{\sigma_2} \int_0^\infty \tilde{h}_{\sigma_2}(s) e^{-\epsilon_{\sigma_2} s} ds < \epsilon_{\sigma_2}.$$

Then switched delay systems (5.1) are

(1) stable, if  $\exists T > 0$ , such that

$$\frac{\sum_{\sigma_i \in \mathbb{N}_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \sum_{\sigma_2 \in \mathbb{N}_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \leq 1, \quad t \geq T \quad (5.12)$$

and

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \mathbb{N}_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \sum_{\sigma_2 \in \mathbb{N}_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} = 1; \quad (5.13)$$

(2) locally exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \mathbb{N}_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} k_{\sigma_1}(t) \tau + \sum_{\sigma_2 \in \mathbb{N}_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \mathbb{N}_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} < 1. \quad (5.14)$$

The proof is similar to that of Theorem 5.1.

As in Remark 1, in Theorems 5.1 and 5.2, (5.2)–(5.4) and (5.12)–(5.14) are actually switching laws which guarantee the switched delay system (5.1) to be stable or locally exponentially stable. We give an example to show how to design switching laws for the switched delay systems.

**Example 5.1.** Consider nonlinear switched delay systems

$$x'(t) = A_i x + f_i(t, x_t) + \int_{t_0}^t h_i(t-s, x(s)) ds, \quad i = 1, 2 \quad (5.15)$$

where  $x = (x_1, x_2)^T \in R^2$ ,

$$\begin{aligned} A_1 &= \begin{bmatrix} -4 & 1 \\ -1 & -5 \end{bmatrix}, & f_1(t, x_t) &= \begin{bmatrix} 0 \\ x_1^3(t-1) \end{bmatrix}, & h_1(t-s, x(s)) &= \begin{bmatrix} e^{-2(t-s)} x_2^3(s) \\ 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, & f_2(t, x_t) &= \begin{bmatrix} \sin x_1(t-1) \\ \sin x_2(t-1) \end{bmatrix}, & h_2(t-s, x(s)) &= \begin{bmatrix} e^{t-s} x_1 \\ e^{t-s} x_2 \end{bmatrix}. \end{aligned}$$

With Theorem 5.2, we have  $\lambda_1 = -4$ ,  $\lambda_2 = 2$ ;  $\tilde{f}_1(\|x_t\|) = \|x_t\|^2$ ,  $b_2 = 1$ ;  $\tilde{h}_1(t-s, x(s)) = e^{-2(t-s)} \|x(s)\|^2$ ,  $\tilde{h}_2(t) = e^t$ . Take  $\tau = 1$ ,  $\epsilon_1 = 1/2$ ,  $\epsilon_2 = 7/2$ ,  $M_1 = M_2 = 1$ ,  $r = 1$ .

Then,

$$\epsilon_1 + M_1 \tilde{f}_1(r) e^{\epsilon_1 \tau} + M_1 \int_0^\infty \tilde{h}_1(s, r) e^{\epsilon_1 s} ds < 4 = \lambda_1,$$

$$\lambda_2 + M_2 b_2 + M_2 \int_0^\infty \tilde{h}_2(s) e^{-\epsilon_2 s} ds < 7/2 = \epsilon_2.$$

The switching laws are:

- (S<sub>1</sub>)\*  $\pi_1(t) \geq k_1(t) + 7\pi_2(t)$  and  $\limsup_{t \rightarrow \infty} \frac{k_1(t) + 7\pi_2(t)}{\pi_1(t)} = 1$ .  
 (S<sub>2</sub>)\*  $\limsup_{t \rightarrow \infty} \frac{k_1(t) + 7\pi_2(t)}{\pi_1(t)} < 1$ .

By Theorem 5.2,

- if we choose switching law (S<sub>1</sub>)\*, then the switched delay system

$$x'(t) = A_i x(t) + f_i(t, x_t) + \int_{t_0}^t h_i(t-s, x(s)) ds, \quad (5.16)$$

where  $i = 1, 2, 0 \leq \tau(t) \leq 1$ , is stable;

- if we choose switching law (S<sub>2</sub>)\*, then the switched delay system (5.16) is locally exponentially stable.

**Remark 2.** (a) If  $r = \infty$  in Theorems 5.1 and 5.2, then the results are globally exponential stability.

- (b) If in some  $\sigma_1$ th subsystem,  $\lambda_{\sigma_1}$  is not large enough to satisfy the condition (ii) in Theorems 5.1 and 5.2, we can consider it as a  $\sigma_2$  subsystem and use a similar approach to discuss switching laws so that the switched delay system is stable.  
 (c) If  $\aleph_{\sigma_1} = \{1\}$  and  $f_{\sigma_2} \equiv 0$ , i.e., the switched system is composed by a stable delay system and some unstable non-delay systems, then we can obtain some more simple criteria.

**Corollary 5.1.** Assume that all the conditions of Theorems 5.1 or 5.2 hold except  $\tilde{f}_{\sigma_2} \equiv 0$  and  $\aleph_1 = \{1\}$ . Then switched system (5.1) is

- (1) stable, if  $\exists T > 0$ , such that

$$\frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} \leq 1, \quad t \geq T$$

and

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} = 1;$$

- (2) locally exponentially stable, if

$$\limsup_{t \rightarrow \infty} \frac{\sum_{\sigma_i \in \aleph_i} k_{\sigma_i}(t) \ln M_{\sigma_i} + \sum_{\sigma_2 \in \aleph_2} \epsilon_{\sigma_2} \pi_{\sigma_2}(t)}{\sum_{\sigma_1 \in \aleph_1} \epsilon_{\sigma_1} \pi_{\sigma_1}(t)} < 1.$$

## 6. Conclusion

We have studied some stability properties of linear and nonlinear switched delay systems consisting of both stable and unstable subsystems. Some switching laws have been developed, which show that if the average of total activation time of unstable subsystems is relatively small compared with that of stable subsystems, then global exponential stability for linear switched delay systems, and local exponential stability for nonlinear switched delay systems can be guaranteed. In the interests of brevity, only single delay is studied in this paper, but it can be extended to the case with multiple delays.

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